

# Resolution-Based Uniform Interpolation for Expressive Description Logics

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# Uniform Interpolation

- Restrict ontology  $\mathcal{O}$  to a set of given symbols  $\mathcal{S}$
- Preserve entailments in  $\mathcal{S}$
- Forget symbols outside  $\mathcal{S}$

## Input Ontology

Margherita  $\sqsubseteq \forall \text{topping}. (\text{Tomato} \sqcup \text{Mozarella})$   
American  $\sqsubseteq \exists \text{topping}. \text{Tomato}$   
American  $\sqsubseteq \exists \text{topping}. \text{Mozarella}$   
American  $\sqsubseteq \exists \text{topping}. \text{Pepperoni}$   
 $\text{Tomato} \sqcup \text{Mozarella} \sqsubseteq \text{VegTopping}$   
 $\text{Pepperoni} \sqsubseteq \text{MeatTopping}$



## Uniform Interpolant

Margherita  $\sqsubseteq \forall \text{topping}. \text{VegTopping}$   
American  $\sqsubseteq \exists \text{topping}. \text{MeatTopping}$

# Applications

- Exhibit hidden concept relations
- Compare ontology versions (logical difference)
- Information Hiding
- Obfuscation
- ...

# Expressive Description Logics

## Concepts $\mathcal{ALC}$

$$\perp \mid \top \mid A \mid \neg C \mid C \sqcup D \mid C \sqcap D \mid \exists r.C \mid \forall r.C$$

## TBox Axioms $\mathcal{ALC}$

$$C \sqsubseteq D \mid C \equiv D$$

## ABox Axioms $\mathcal{ALC}$

$$C(a) \mid r(a, b)$$

$\mathcal{ALCH}$ : Role Hierarchies

 $r \sqsubseteq s$ 

$\mathcal{SH}$ : Transitive Roles

 $trans(r)$ 

$\mathcal{SHQ}$ : Number Restrictions

 $\geq nr.C, \leq nr.C$ 

$\mathcal{SHI}$ : Inverse Roles

 $r^{-1}$ 

$\mathcal{L}\mu$ : Fixpoint Operators

 $\mu X.C[X], \nu X.C[X]$

# Uniform Interpolants

## Definition

$\mathcal{O}^{\mathcal{S}}$  is an  $\mathcal{L}$  uniform interpolant (UI) of  $\mathcal{O}$  for  $\mathcal{S}$  iff

1.  $\text{sig}(\mathcal{O}^{\mathcal{S}}) \subseteq \mathcal{S}$
2.  $\mathcal{O}^{\mathcal{S}} \models \alpha$  iff  $\mathcal{O} \models \alpha$  where
  - $\text{sig}(\alpha) \subseteq \mathcal{S}$
  - $\alpha$  is expressible in  $\mathcal{L}$

# Existing Work

Description Logic	Publication
DL-Lite	Wang, Wang et. al 2008
$\mathcal{EL}$	Konev, Walther, et. al 2009 Nikitina 2011 Lutz, Seylan 2012
$\mathcal{ALC}$	Lutz, Wolter 2011 (Foundations) Wang, Wang et. al 2012 (Tableau-based) Ludwig, Konev 2013 (Resolution-based) Koopmann, Schmidt 2013 (Res.+Fixpoints)
$\mathcal{ALCH}$	Koopmann, Schmidt 2013 (Role Forgetting)
$\mathcal{SHQ}$	Koopmann, Schmidt 2014 (Res.+Fixpoints)

# Challenges: Finiteness

UIs in input language not always finite

- Example  $\mathcal{ALC}$ :

$$\begin{aligned} A \sqsubseteq B, \quad B \sqsubseteq \exists r.B \\ \mathcal{S} = \{A, r\} \end{aligned}$$

- Uniform Interpolant in  $\mathcal{ALC}$ :

- $A \sqsubseteq \exists r. \dots$

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- Uniform Interpolant in  $\mathcal{ALC}$ :

$$- A \sqsubseteq \exists r. \dots$$

- Solutions:

Fixpoints:  $A \sqsubseteq \nu X. (\exists r. X)$

Approximate:  $A \sqsubseteq \exists r. \exists r. \exists r. \top$

Helper concepts:  $A \sqsubseteq \exists r. D, \quad D \sqsubseteq \exists r. D$

# Challenges: Complexity

Known complexities  $\mathcal{ALC}$ :

- If finite, worst size  $O(2^{2^n})$   
    ⇒ With fixpoints:  $O(2^{2^n})$
- Deciding finiteness:  $O(2^{2^n})$

# Challenges: Reasoning

Using reasoning techniques for uniform interpolation

- Might derive not enough
  - Often optimised for specific problem
  - Represent as  $\mathcal{L}$  ontology
- Might derive too much
  - Only entailments in  $\mathcal{L}$  needed
  - Termination
  - Goal-oriented inferences necessary

⇒ Need for new calculi

# Solutions

Representation:

- Reason on  $\mathcal{L}$ -statements
  - Only infer entailments in  $\mathcal{L}$
  - ⇒ Clauses = DL axioms
  - ⇒ Cheap conversion of result
- Use finitely bounded representation
  - Ensure termination
  - Preserve all entailments
  - ⇒ Structural transformation does the job

## Normal form, $\mathcal{ALC}$

### $\mathcal{ALC}$ -Clause

$$T \sqsubseteq L_1 \sqcup \dots \sqcup L_n$$

$L_i$ :  $\mathcal{ALC}$ -literal

### $\mathcal{ALC}$ -Literal

$$A \mid \neg A \mid \exists r.D \mid \forall r.D$$

$A$ : any concept symbol,  $D$ : definer symbol

- Definer symbols: Special concept symbols, not part of signature
- Invariant: max 1 neg. definer symbol per clause  
 $\Rightarrow \neg D_1 \sqcup \exists r.D_2 \sqcup \neg B, \quad \underline{\neg D_1 \sqcup \neg D_2 \sqcup A}$

## Definer symbols

Invariant: max 1 neg. definer symbol per clause

- Allows easy translation to clausal form and back:

$$C_1 \sqcup Qr.C_2 \iff C_1 \sqcup Qr.D_1, \neg D_1 \sqcup C_2$$

$$C_1 \sqcup \nu X. C_2[X] \iff C_1 \sqcup Qr.D_1, \neg D_1 \sqcup C_2[D]$$

$\Rightarrow$  Any set of clauses can be converted into an  $\mathcal{ALC}\mu$ -ontology  
( $\mathcal{ALC}$  with fixpoints)

- New definer symbols introduced by calculus
  - Number finitely bounded

# Calculus

## Resolution + Combination rules

- Resolution rule:
  - Direct inference on concept symbol to forget
  - Resolvent has to fulfil invariant

$$\frac{C_1 \sqcup A \quad C_2 \sqcup \neg A}{C_1 \sqcup C_2}$$

# Calculus

## Resolution + Combination rules

- Resolution rule:

- Direct inference on concept symbol to forget
- Resolvent has to fulfil invariant

$$\frac{C_1 \sqcup A \quad C_2 \sqcup \neg A}{C_1 \sqcup C_2}$$

- Combination rules:

- Combine contexts of nested definer symbols
- Introduce new definer symbols
  - Representing conjunctions of definers
  - Max.  $2^n$  new definer symbols
- Make further inferences possible

# Combination Rules

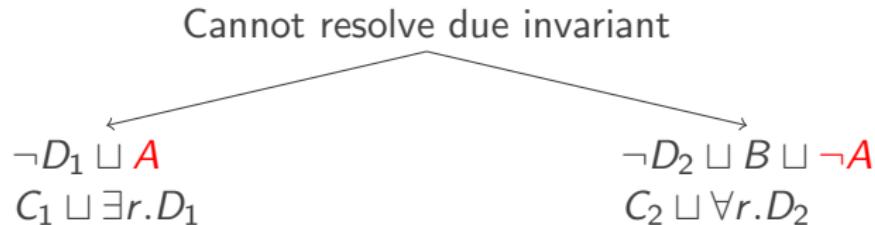
$$\begin{array}{l} \neg D_1 \sqcup A \\ C_1 \sqcup \exists r.D_1 \end{array}$$

$$\begin{array}{l} \neg D_2 \sqcup B \sqcup \neg A \\ C_2 \sqcup \forall r.D_2 \end{array}$$

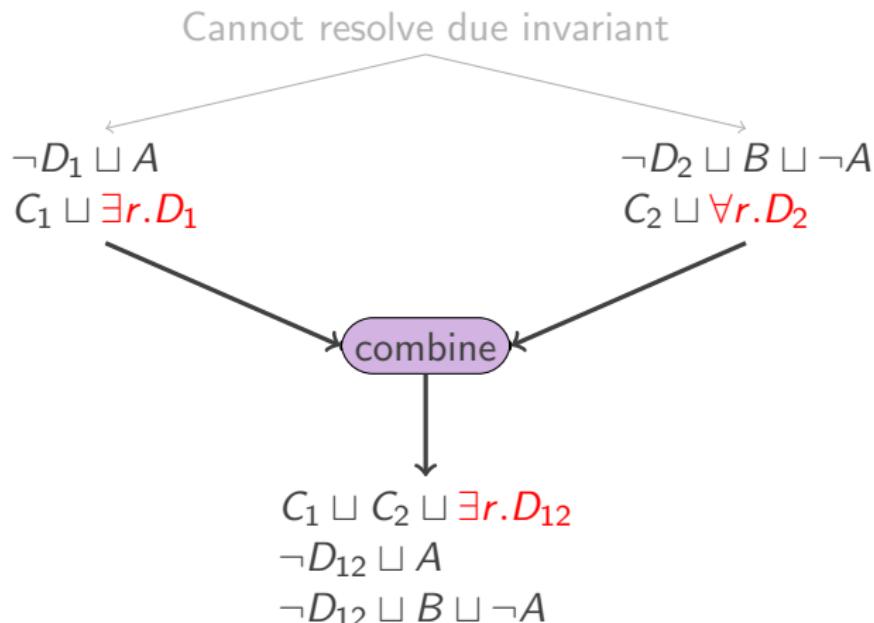
$$T \sqsubseteq C_1 \sqcup \exists r.A$$

$$T \sqsubseteq C_2 \sqcup \forall r.(B \sqcup \neg A)$$

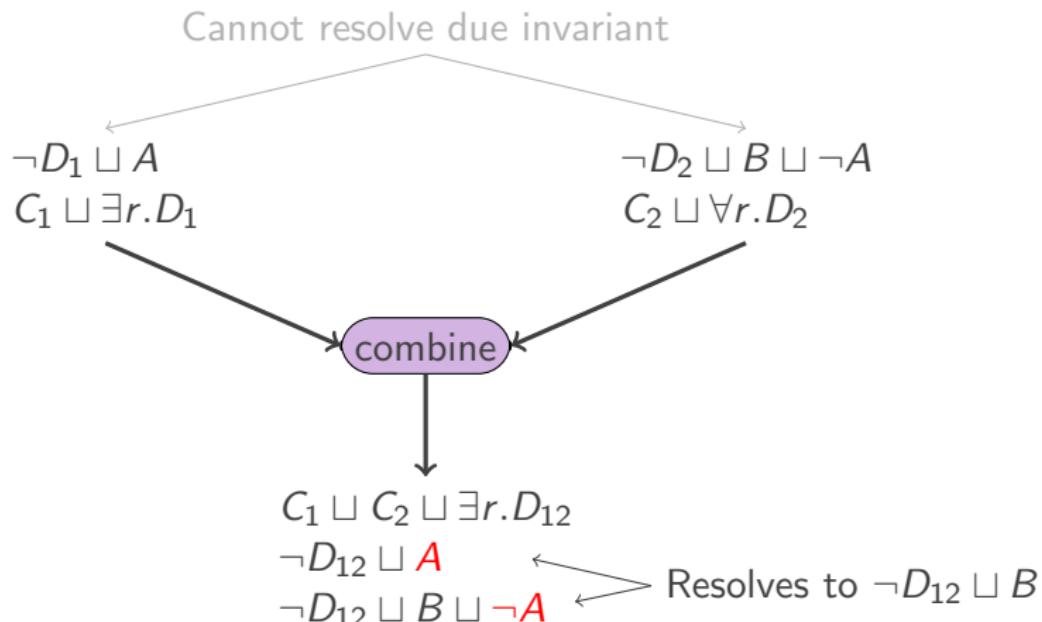
# Combination Rules



# Combination Rules



# Combination Rules



# Combination Rules $\mathcal{ALC}$

$\forall\exists$ -Combination

$$\frac{C_1 \sqcup \forall r.D_1 \quad C_2 \sqcup \exists r.D_2}{C_1 \sqcup C_2 \sqcup \exists r.D_{12}}$$

$\forall\forall$ -Combination

$$\frac{C_1 \sqcup \forall r.D_1 \quad C_2 \sqcup \forall r.D_2}{C_1 \sqcup C_2 \sqcup \forall r.D_{12}}$$

# Combination Rules $\mathcal{SHQ}$

## $\leq\leq$ -Combination:

$$\frac{C_1 \sqcup \leq n_1 r_1. \neg D_1 \quad C_2 \sqcup \leq n_2 r_2. \neg D_2 \quad r \sqsubseteq r_1 \quad r \sqsubseteq r_2}{C_1 \sqcup C_2 \sqcup \leq (n_1 + n_2) r. \neg D_{12}}$$

## $\geq\leq$ -Combination:

$$\frac{C_1 \sqcup \geq n_1 r_1. (D_1 \sqcup \dots \sqcup D_m) \quad C_2 \sqcup \leq n_2 r_2. \neg D_a \quad r_1 \sqsubseteq_R r_2}{C_1 \sqcup C_2 \sqcup \geq (n_1 - n_2) r_1. (D_{1,a} \sqcup \dots \sqcup D_{m,a})}$$

## $\leq\geq$ -Combination:

$$\frac{C_1 \sqcup \leq n_1 r_1. \neg D_1 \quad C_2 \sqcup \geq n_2 r_2. D_2 \quad r_2 \sqsubseteq_R r_1 \quad n_1 \geq n_2}{C_1 \sqcup C_2 \sqcup \leq (n_1 - n_2) r_1. \neg (D_1 \sqcup D_2) \sqcup \geq 1 r_1. D_{12}}$$

⋮

$$C_1 \sqcup C_2 \sqcup \leq (n_1 - 1) r_1. \neg (D_1 \sqcup D_2) \sqcup \geq n_2 r_1. D_{12}$$

## $\geq\geq$ -Combination:

$$\frac{C_1 \sqcup \geq n_1 r_1. D_1 \quad C_2 \sqcup \geq n_2 r_2. D_2 \quad r_1 \sqsubseteq_R r \quad r_2 \sqsubseteq_R r}{C_1 \sqcup C_2 \sqcup \geq (n_1 + n_2) r. (D_1 \sqcup D_2) \sqcup \geq 1 r. D_{12}}$$

⋮

$$C_1 \sqcup C_2 \sqcup \geq (n_1 + 1) r. (D_1 \sqcup D_2) \sqcup \geq n_2 r. D_{12}$$

## Transitivity:

$$\frac{C \sqcup \leq 0 r_1. \neg D \quad \text{trans}(r_2) \in \mathcal{R} \quad r_2 \sqsubseteq_R r_1}{C \sqcup \leq 0 r_2. \neg D' \quad \neg D' \sqcup D \quad \neg D' \sqcup \leq 0 r_2. \neg D'}$$

# Algorithm

**INPUT:** Ontology  $\mathcal{O}$ , signature  $\mathcal{S}$

**OUTPUT:** Uniform interpolant of  $\mathcal{O}$  for  $\mathcal{S}$

1. Transform  $\mathcal{O}$  to normal form  $\mathbf{N}$
2. For each  $x \in \text{sig}(\mathcal{O}) \setminus \mathcal{S}$ :
  - 2.1 Derive all inferences on  $x$
  - 2.2 Remove clauses containing  $x$
3. Transform  $\mathbf{N}$  to ontology  $\mathcal{O}^{\mathcal{S}}$   
(eliminate definer symbols)

# Empirical Results $\mathcal{ALCH}$

$ \mathcal{S} $	Timeouts	Fixpoints	Interpolant Size	Axiom Size	Average Duration
50	15.12%	6.99%	22.50%	799.37%	24.2 sec.
100	18.38%	11.57%	45.21%	646.32%	21.0 sec.
150	22.25%	13.58%	76.55%	837.66%	23.7 sec.
All	18.38%	10.44%	45.74%	757.69%	23.0 sec.

- 115 ontologies from NCBO BioPortal  
 $\Rightarrow \mathcal{ALCH}$ -fragments
- Timeout: 1,000 seconds

# Summary + Outlook

- Method implemented for:
  - $\mathcal{ALC}$ -ontologies with ABoxes
  - $\mathcal{ALCH}$ -TBoxes
  - $\mathcal{SHQ}$ -TBoxes
  
- Future work:
  - ABox support for all approaches
  - $\mathcal{SHI}$  (inverse roles)
  - $\mathcal{SHIQ}$