Self-Extensionality of Finitely-Valued Logics

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March 12, 2021

SELF-EXTENSIONALITY OF FINITELY-VALUED LOGICS

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Abstract. We start from proving a general characterization of the self-extensionality of sentential logics implying the decidability of this problem as for (possibly, multiple) finitely-valued logics. And what is more, in case of finitely-/three-valued logics with "equality determinant as well as classical either implication or both conjunction and disjunction"/"classical conjunction and subclassical negation", we then derive a characterization yielding a quite effective algebraic criterion of checking their self-extensionality via analyzing homomorphisms between (viz., in the unitary case, endomorphisms of) their underlying algebras and equally being a quite useful heuristic tool, manual applications of which are demonstrated within the framework of Łukasiewicz' finitely-valued logics, four-valued expansions of Belnap's "useful" four-valued logic, their non-unitary three-valued extensions, unitary inferentially consistent non-classical ones being well-known to be non-self-extensional, as well as unitary three-valued implicative/"[both] conjunctive [and disjunctive]" logics with subclassical negation (including both paraconsistent and paracomplete ones).

1. Introduction

Recall that a sentential logic (cf., e.g., [8]) is said to be *self-extensional*, whenever its inter-derivability relation is a congruence of the formula algebra. Such feature is typical of both two-valued (in particular, classical) and super-intuitionistic logics as well as some interesting many-valued ones (like Belnap's "useful" four-valued one [3]). Here, we explore it laying a special emphasis onto the general framework of finitely-valued logics and the decidability issue with reducing the complexity of effective procedures of verifying it, when restricting our consideration by those logics of such a kind which possess certain peculiarities — both classical either implication or both conjunction and disjunction (in Tarski's conventional sense) and binary equality determinant in a sense extending [21] towards [22]. We then exemplify our universal elaboration by discussing four (perhaps, most representative) generic classes of logics of the kind involved: Łukasiewicz' finitely-valued logics [9], four-valued expansions of Belnap's logic (cf. [19]), their non-unitary three-valued extensions, unitary inferentially consistent non-classical ones being well-known (due to [22]) to be non-self-extensional, as well as unitary three-valued implicative/"[both] conjunctive [and disjunctive]" logics with subclassical negation (including both paraconsistent and paracomplete ones).

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set and Lattice Theory, Universal Algebra and Logic to be found, if necessary, in standard mathematical handbooks like [2, 5, 12]). Section 2 is a concise summary of particular basic issues underlying the paper, most of which, though having become a part of algebraic and logical folklore, are still recalled just for the exposition to be properly self-contained. In Section 3, we then develop/recall certain advanced

2020 *Mathematics Subject Classification.* 03B20, 03B22, 03B50, 03B53, 03G10.
Key words and phrases. logic; calculus; matrix; extension.

generic issues concerning false-singular consistent weakly conjunctive matrices, disjunctivity, implicativity and equality determinants as well as classical matrices and logics. Section 5 is a collection of main *general* results of the paper that are then exemplified in Section 6 (aside from Łukasiewicz' finitely-valued logics, whose non-self-extensionality has actually been due [22], as we briefly discuss within Example 5.16 — this equally concerns certain particular instances discussed in Section 6 and summarized in Example 5.17). Finally, Section 7 is a brief summary of principal contributions of the paper.

2. Basic issues

Notations like img, dom, ker, hom, π_i and Con and related notions are supposed to be clear.

2.1. **Set-theoretical background.** We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. Then, given any $(N \cup \{n\}) \subseteq \omega$, set $(N \div n) \triangleq \{\frac{m}{n} \mid m \in N\}$. The proper class of all ordinals is denoted by ∞. Also, functions are viewed as binary relations, while singletons are identified with their unique elements, unless any confusion is possible. A function f is said to be *singular*, provided $|\operatorname{img} f| \in 2$.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] (including a set T) is denoted by $\wp_{[K]}((T,)S)$, respectively. Then, an *enumeration of* S is any bijection from $|S|$ onto S. As usual, given any equivalence relation θ on S, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq \theta[\{a\}]$, for all $a \in S$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in the either sequence $\bar{t}$ or vector $\vec{t}$ form, its s-th component (viz., the value under argument s), where $s \in S$, being written as either t_s or t^s, respectively. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp., mapping $R : A^S \to B^S$) point-wise. Likewise, given a set A, an S-tuple $\overline{B}$ of sets and any $\bar{f} \in (\prod_{s \in S} B_s^A)$, put $(\prod \bar{f}) : A \to (\prod \overline{B}), a \mapsto \langle f_s(a)\rangle_{s \in S}$. (In case $I = 2$, $f_0 \times f_1$ stands for $(\prod \bar{f})$.) Further, set $\Delta_S \triangleq \{\langle a, a\rangle \mid a \in S\}$, functions of such a kind being referred to as *diagonal*, and $S^+ \triangleq \bigcup_{i \in (\omega \setminus 1)} S^i$, elements of $S^* \triangleq (S^0 \cup S^+)$ being identified with ordinary finite tuples/sequences, the binary concatenation operation on which being denoted by $*$, as usual. Then, any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = (\operatorname{dom} \bar{a})$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}\restriction(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

In particular, given any $f : S \to S$ and any $n \in \omega$, set $f^n \triangleq (\circ\langle n \times \{f\}, \Delta_S\rangle) : S \to S$. Likewise, given a one more set T, any $\diamond : (S \times T) \to T$ determines the equally-denoted mapping $\diamond : (S^* \times T) \to T$ as follows: by induction on the length (viz., domain) l of any $\bar{a} \in S^*$, for all $b \in T$, put:

$$(\bar{a} \diamond b) \triangleq \begin{cases} b & \text{if } l = 0, \\ a_0 \diamond (((\bar{a}\restriction(l \setminus 1)) \circ ((+1)\restriction(l-1))) \diamond b) & \text{otherwise.} \end{cases}$$

Finally, given any $T \subseteq S$, we have the *characteristic function* $\chi_S^T \triangleq ((T \times \{1\}) \cup ((S \setminus T) \times \{0\}))$ *of* T *in* S.

Let A be a set. Then, an $X \in S \subseteq \wp(A)$ is said to be *meet-irreducible in/of* S, provided, for each $T \in \wp(S)$, $X \in T$, whenever $T = (A \cap \bigcap T)$, the set of all them being denoted by $\mathrm{MI}(S)$. Next, a $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$, in which case $U \neq \varnothing$, when taking $S = \varnothing$. Next, a subset of $\wp(A)$ is said to be *inductive*, whenever it is closed under unions of upward-directed subsets. Further, a *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$. In that case, any $\mathcal{B} \subseteq \mathcal{C}$ is called a *(closure) basis of* $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. Furthermore, an *operator over* A is any unary operation O on $\wp(A)$. This is said to be *(monotonic) [idempotent]* $\{$*transitive*$\}$ $\langle$*inductive/finitary/compact*$\rangle$, provided, for all $(B,)D \in \wp(A)$ $\langle$resp., any upward-directed $U \subseteq \wp(A)\rangle$, it holds that $(O(B))[D]\{O(O(D)\} \subseteq O(D)\langle O(\bigcup U) \subseteq \bigcup O[U]\rangle$. Finally, a *closure operator over* A is any monotonic idempotent transitive operator over A, in which case $\operatorname{img} C$ is a closure system over A, determining C uniquely, because, for every closure basis $\mathcal{B}$ of $\operatorname{img} C$ (including $\operatorname{img} C$ itself) and each $X \subseteq A$, it holds that $C(X) = (A \cap \bigcap \{Y \in \mathcal{B} | X \subseteq Y\})$, called *dual to* C and vice versa. (Clearly, C is inductive iff $\operatorname{img} C$ is so.)

Remark 2.1. By Zorn Lemma, due to which any non-empty inductive subset of $\wp(A)$ has a maximal element, $\mathrm{MI}(\mathcal{C})$ is a basis of any inductive closure system $\mathcal{C}$ over A. □

2.2. **Algebraic background.** Unless otherwise specified, abstract algebras are denoted by Fraktur letters [possibly, with indices], their carriers (viz., underlying sets) being denoted by corresponding Italic letters [with same indices, if any].

A *(propositional/sentential) language/signature* is any algebraic (viz., functional) signature Σ (to be dealt with throughout the paper by default) constituted by function (viz., operation) symbols of finite arity to be treated as *(propositional/sentential) connectives.*

Given a Σ-algebra $\mathfrak{A}$, $\mathrm{Con}(\mathfrak{A})$ is an inductive closure system over A^2 forming a bounded lattice with meet $\theta \cap \vartheta$ of any $\theta, \theta \in \mathrm{Con}(\mathfrak{A})$, their join $\theta \amalg \vartheta$, being the transitive closure of $\theta \cup \vartheta$, zero Δ_A and unit A^2, the dual closure operator being denoted by $\mathrm{Cg}^{\mathfrak{A}}$. Then, $\mathfrak{A}$ is said to be *simple*, provided the lattice involved is two-element, in which case $|A| > 1$. Next, a *[partial] endomorphism of* $\mathfrak{A}$ is any homomorphism from [a subalgebra of] $\mathfrak{A}$ to $\mathfrak{A}$. Further, a $B \subseteq A$ is said to *"form a subalgebra of* $\mathfrak{A}$*"/"be* $\mathfrak{A}$*-closed"*, whenever it is closed under operations of $\mathfrak{A}$. Furthermore, given a class K of Σ-algebras, set $\hom(\mathfrak{A}, \mathsf{K}) \triangleq (\bigcup \{\hom(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}\})$, in which case $\ker[\hom(\mathfrak{A}, \mathsf{K})] \subseteq \mathrm{Con}(\mathfrak{A})$, and so $(A^2 \cap \bigcap \ker[\hom(\mathfrak{A}, \mathsf{K})]) \in \mathrm{Con}(\mathfrak{A})$.

Given any $\alpha \in \wp(\omega)$, put $\bar{x}_\alpha \triangleq \langle x_\beta \rangle_{\beta \in \alpha}$, $V_\alpha \triangleq (\operatorname{img} \bar{x}_\alpha)$, elements of which being viewed as *(propositional/sentential) variables of rank* α, and $((\forall|\exists)_\alpha) \triangleq ((\forall|\exists)\bar{x}_\alpha)$. Then, providing either $\alpha \neq \varnothing$ or Σ has a nullary symbol, we have the absolutely-free Σ-algebra $\mathfrak{Fm}^\alpha_\Sigma$ freely-generated by the set V_α, its endomorphisms/elements of its carrier $\mathrm{Fm}^\alpha_\Sigma$ being called *(propositional/sentential)* Σ*-substitutions/-formulas of rank* α. A $\theta \in \mathrm{Con}(\mathfrak{Fm}^\alpha_\Sigma)$ is said to be *fully invariant*, if, for every Σ-substitution σ of rank α, it holds that $\sigma[\theta] \subseteq \theta$. Recall that

$$\forall h \in \hom(\mathfrak{A}, \mathfrak{B}) : [(\operatorname{img} h) = B) \Rightarrow] \\ (\hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{B}) \supseteq [=] \{h \circ g \mid g \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})\}), \quad (2.1)$$

where $\mathfrak{A}$ and $\mathfrak{B}$ are Σ-algebras. Any $\langle \phi, \psi \rangle \in \mathrm{Eq}^\alpha_\Sigma \triangleq (\mathrm{Fm}^\alpha_\Sigma)^2$ is referred to as a Σ*-equation/-indentity of rank* α and normally written in the standard equational form $\phi \approx \psi$. (In general, any mention of α is normally omitted, whenever $\alpha = \omega$.) In this way, given any $h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})$, $\ker h$ is the set of all Σ-identities of rank α *true/satisfied in* $\mathfrak{A}$ *under* h. Likewise, given a class K of Σ-algebras, $\theta^\alpha_{\mathsf{K}} \triangleq (\mathrm{Eq}^\alpha_\Sigma \cap \bigcap \ker[\hom(\mathfrak{Fm}^\alpha_\Sigma, \mathsf{K})]) \in \mathrm{Con}(\mathfrak{Fm}^\alpha_\Sigma)$, being fully invariant, in view of (2.1), is the set of all all Σ-identities of rank α *true/satisfied in* K, in which case we set $\mathfrak{F}^{\alpha}_{\mathsf{K}} \triangleq (\mathfrak{Fm}^{\alpha}_{\Sigma}/\theta^{\alpha}_{\mathsf{K}})$. (In case both α as well as both K and all members of it are finite, the set $I \triangleq \{\langle h, \mathfrak{A}\rangle \mid h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A}), \mathfrak{A} \in \mathsf{K}\}$ is finite — more precisely, $|I| = \sum_{\mathfrak{A}\in\mathsf{K}} |A|^{\alpha}$, in which case $g \triangleq (\prod_{i\in I} \pi_0(i)) \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \prod_{i\in I}(\pi_1(i){\restriction}\operatorname{img}\pi_0(i)))$ with $(\ker g) = \theta \triangleq \theta^{\alpha}_{\mathsf{K}}$, and so, by the Homomorphism Theorem, $e \triangleq (g \circ \nu_{\theta}^{-1})$ is an isomorphism from $\mathfrak{F}^{\alpha}_{\mathsf{K}}$ onto the subdirect product $(\prod_{i\in I}(\pi_1(i){\restriction}\operatorname{img}\pi_0(i))){\restriction}(\operatorname{img} g)$ of $\langle \pi_1(i){\restriction}\operatorname{img}\pi_0(i)\rangle_{i\in I}$. In this way, the former is finite, for the latter is so — more precisely, $|F^{\alpha}_{\mathsf{K}}| \leqslant (\max_{\mathfrak{A}\in\mathsf{K}} |A|)^{|I|}$.)

A *"congruence-permutation term"/discriminator for* K is any $\tau \in \mathrm{Fm}^3_{\Sigma}$ such that, for each $\mathfrak{A}$ and all $\bar{a} \in A^{2/3}$, it holds that $[\tau^{\mathfrak{A}}(a_0, a_1, a_{1/2}) =]a_0 = \tau^{\mathfrak{A}}(a_1, a_1, a_0)$ [unless $a_0 = a_1$], in which case it is so for any homomorphic image of any subalgebra of $\mathfrak{A}$ /as well as a congruence-permutation term for $\mathfrak{A}$ (when taking $a_2 = a_1$), while, for any $\theta \in \mathrm{Con}(\mathfrak{A})$, any $\langle a, b\rangle \in (\theta \setminus \Delta_A)$ and any $c \in A$, we have $a = \tau^{\mathfrak{A}}(a, b, c)\ \theta\ \tau^{\mathfrak{A}}(a, a, c) = c$, in which case we get $\theta = A^2$, and so $\mathfrak{A}$ is simple, unless it is one-element. By [10] and Lemma 2.10 of [24], we have:

Lemma 2.2. *Let $n \in (\omega[\setminus 1])$, $\overline{\mathfrak{A}}$ an n-tuple of simple Σ-algebras and τ a congruence-permutation term for* $\operatorname{img}\overline{\mathfrak{A}}$. *Then, any subdirect product of $\overline{\mathfrak{A}}$ is isomorphic to the direct product of some [non-empty] subset of $\overline{\mathfrak{A}}$.*

The class of all Σ-algebras satisfying every element of an $\mathcal{E} \subseteq \mathrm{Eq}^{\omega}_{\Sigma}$ is called the *variety axiomatized by* $\mathcal{E}$. Then, the variety $\mathbf{V}(\mathsf{K})$ axiomatized by $\theta^{\omega}_{\mathsf{K}}$ is the least variety including K and is said to be *generated by* K, in which case $\theta^{\alpha}_{\mathbf{V}(\mathsf{K})} = \theta^{\alpha}_{\mathsf{K}}$, and so $\mathfrak{F}^{\alpha}_{\mathbf{V}(\mathsf{K})} = \mathfrak{F}^{\alpha}_{\mathsf{K}}$.

Given a fully invariant $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, by (2.1), $\mathfrak{Fm}^{\omega}_{\Sigma}/\theta$ belongs to the variety V axiomatized by θ, in which case any Σ-identity satisfied in V belongs to θ, and so $\theta^{\omega}_{\mathsf{V}} = \theta$. In particular, given a variety V of Σ-algebras, we have $\mathfrak{F}^{\alpha}_{\mathsf{V}} \in \mathsf{V}$. And what is more, given any $\mathfrak{A} \in \mathsf{V}$ and any $h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$, as $\theta \triangleq \theta^{\alpha}_{\mathsf{V}} \subseteq (\ker h)$, by the Homomorphism Theorem, $g \triangleq (h \circ \nu_{\theta}^{-1}) \in \hom(\mathfrak{F}^{\alpha}_{\mathsf{V}}, \mathfrak{A})$, in which case $h = (g \circ \nu_{\theta})$, and so $\mathfrak{F}^{\alpha}_{\mathsf{V}}$ is a *free algebra of* V *with* $|\alpha|$ *free generators*, whenever V contains a non-one-element member, in which case $\nu_{\theta}{\restriction}V_{\alpha}$ is injective, and so $|\alpha|$ is the cardinality of the set V_{α}/θ generating $\mathfrak{F}^{\alpha}_{\mathsf{V}}$, for V_{α} generates $\mathfrak{Fm}^{\alpha}_{\Sigma}$.

The mapping $\mathrm{Var} : \mathrm{Fm}^{\omega}_{\Sigma} \to \wp_{\omega}(V_{\omega})$ assigning the set of all *actually* occurring variables is defined in the standard recursive manner by induction on construction of Σ-formulas. Given any $(m,)n \in \omega$, the Σ-substitution extending $(\Delta_{V_m}\cup)[x_i/x_{i+n}]_{i\in(\omega(\setminus m))}$ is denoted by $\sigma_{(m:)+n}$.

2.2.1. *Equational disjunctive and implicative systems.* According to [22, 24], a(n) *(equational) disjunctive/implicative system for* a class K of Σ-algebras is any $\mho \subseteq \mathrm{Eq}^4_{\Sigma}$ such that, for each $\mathfrak{A} \in \mathsf{K}$ and all $\bar{a} \in A^4$, it holds that:

$$((a_0 \neq / = a_1) \Rightarrow (a_2 = a_3)) \Leftrightarrow (\mathfrak{A} \models (\textstyle\bigwedge \mho)[x_i/a_i]_{i\in 4}). \tag{2.2}$$

2.2.2. *Lattice-theoretic background.*

2.2.2.1. Semi-lattices. Let $\diamond$ be a (possibly, secondary) binary connective of Σ.

A Σ-algebra $\mathfrak{A}$ is called a $\diamond$-*semi-lattice*, provided it satisfies semilattice (viz., idempotencity, commutativity and associativity) identities for $\diamond$, in which case we have the partial ordering $\leq^{\mathfrak{A}}_{\diamond}$ on A, given by $(a \leq^{\mathfrak{A}}_{\diamond} b) \overset{\text{def}}{\Longleftrightarrow} (a = (a \diamond^{\mathfrak{A}} b))$, for all $a, b \in A$. Then, in case the poset $\langle A, \leq^{\mathfrak{A}}_{\diamond}\rangle$ has the least element (viz., *zero*) [in particular, when A is finite], this is denoted by $\flat^{\mathfrak{A}}_{\diamond}$, while $\mathfrak{A}$ is referred to as a $\diamond$-*semi-lattice with zero (a)* (whenever $a = \flat^{\mathfrak{A}}_{\diamond}$).

Lemma 2.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\diamond$-semi-lattices with zero and $h \in \hom(\mathfrak{A}, \mathfrak{B})$. Suppose $h[A] = B$. Then, $h(\flat^{\mathfrak{A}}_{\diamond}) = \flat^{\mathfrak{B}}_{\diamond}$.*

Proof. Then, there is some $a \in A$ such that $h(a) = \flat_\diamond^{\mathfrak{B}}$, in which case $(a \diamond^{\mathfrak{A}} \flat_\diamond^{\mathfrak{A}}) = \flat_\diamond^{\mathfrak{A}}$, and so $h(\flat_\diamond^{\mathfrak{A}}) = (h(a) \diamond^{\mathfrak{B}} h(\flat_\diamond^{\mathfrak{A}})) = (\flat_\diamond^{\mathfrak{B}} \diamond^{\mathfrak{B}} h(\flat_\diamond^{\mathfrak{A}})) = \flat_\diamond^{\mathfrak{B}}$, as required. □

2.2.2.1.1. Implicative inner semilattices. Set $(x_0 \uplus_\diamond x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$.

A Σ-algebra $\mathfrak{A}$ is called an *$\diamond$-implicative inner semi-lattice*, provided it is a $\uplus_\diamond$-semilattice and satisfies the Σ-identities:

$$(x_0 \diamond x_0) \approx (x_1 \diamond x_1), \tag{2.3}$$
$$((x_0 \diamond x_0) \diamond x_1) \approx x_1, \tag{2.4}$$

in which case it is an $\uplus_\diamond$-semilattice with zero $a \diamond^{\mathfrak{A}} a$, for any $a \in A$.

2.2.2.2. Distributive lattices. Let $\bar{\wedge}$ and $\veebar$ be (possibly, secondary) binary connectives of Σ.

A Σ-algebra $\mathfrak{A}$ is called a *[distributive] $(\bar{\wedge}, \veebar)$-lattice*, provided it satisfies [distributive] lattice identities for $\bar{\wedge}$ and $\veebar$ (viz., semilattice identities for both $\bar{\wedge}$ and $\veebar$ as well as mutual [both] absorption [and distributivity] identities for them), in which case $\leq_{\bar{\wedge}}^{\mathfrak{A}}$ and $\leq_{\veebar}^{\mathfrak{A}}$ are inverse to one another, and so, in case $\mathfrak{A}$ is a $\veebar$-semilattice with zero (in particular, when A is finite), $\flat_{\veebar}^{\mathfrak{A}}$ is the greatest element (viz., *unit*) of the poset $\langle A, \leq_{\bar{\wedge}}^{\mathfrak{A}} \rangle$. Then, in case $\mathfrak{A}$ is a {distributive} $(\bar{\wedge}, \veebar)$-lattice, it is said to be that *with zero/unit (a)*, whenever it is a $(\bar{\wedge}/\veebar)$-semilattice with zero (a).

Let $\Sigma_{+[,01]} \triangleq \{\wedge, \vee[, \bot, \top]\}$ be the [bounded] lattice signature with binary $\wedge$ (conjunction) and $\vee$ (disjunction) [as well as nullary $\bot$ and $\top$ (falsehood/zero and truth/unit constants, respectively)]. Then, a $\Sigma_{+[,01]}$-algebra $\mathfrak{A}$ is called a *[bounded] (distributive) lattice*, whenever it is a (distributive) $(\wedge, \vee)$-lattice [with zero $\bot^{\mathfrak{A}}$ and unit $\top^{\mathfrak{A}}$] {cf., e.g., [2]}.

Given any $n \in (\omega \setminus 2)$, by $\mathfrak{D}_{n[,01]}$ we denote the [bounded] distributive lattice given by the chain $n \div (n-1)$ ordered by $\leqslant$.

2.2.2.2.1. De Morgan lattices. Let $\Sigma_{+,\sim[,01]} \triangleq (\Sigma_{+[,01]} \cup \{\sim\})$ with unary $\sim$ (negation). Then, a *[bounded] De Morgan lattice* [19] is any $\Sigma_{+,\sim[,01]}$-algebra, whose $\Sigma_{+[,01]}$-reduct is a [bounded] distributive lattice and that satisfies the following $\Sigma_{+,\sim}$-identities:

$$\sim\sim x_0 \approx x_0, \tag{2.5}$$
$$\sim(x_0 \wedge x_1) \approx (\sim x_0 \vee \sim x_1), \tag{2.6}$$
$$\sim(x_0 \vee x_1) \approx (\sim x_0 \wedge \sim x_1), \tag{2.7}$$

By $\mathfrak{DM}_{4[,01]}$ we denote the [bounded] De Morgan lattice with $(\mathfrak{DM}_{4[,01]} \restriction \Sigma_{+[,01]}) \triangleq \mathfrak{D}^2_{2[,01]}$ and $\sim^{\mathfrak{DM}_{4[,01]}} \langle i, j \rangle \triangleq \langle 1-j, 1-i \rangle$, for all $i, j \in 2$.

Likewise, given any $n \in (\omega \setminus 2)$, by $\mathfrak{K}_{n[,01]}$ we denote the [bounded] De Morgan lattice with $(\mathfrak{K}_{n[,01]} \restriction \Sigma_{+[,01]}) \triangleq \mathfrak{D}_{n[,01]}$ and $\sim^{\mathfrak{K}_{n[,01]}} k \triangleq (1-k)$, for all $k \in n$, in which case $\mathfrak{K}_{2[,01]}$ is a subalgebra of $\mathfrak{K}_{n[,01]}$. Then, given any $m \in (4 \setminus 2)$, the mapping $e_m : (m \div (m-1)) \to 2^2, a \mapsto \langle [a], (2 \cdot a) - [a] \rangle$ is an embedding of $\mathfrak{K}_{m[,01]}$ into $\mathfrak{DM}_{4[,01]}$.

2.3. **Propositional logics and matrices.** A *[finitary/unary/axiomatic] Σ-rule* is any couple $\langle \Gamma, \varphi \rangle$, where $\Gamma \in \wp_{[\omega/(2\setminus 1)/1]}(\mathrm{Fm}^\omega_\Sigma)$ and $\varphi \in \mathrm{Fm}^\omega_\Sigma$, normally written in the standard sequent form $\Gamma \vdash \varphi$, $\varphi|(\psi \in \Gamma)$ being referred to as the|a *conclusion|premise of* it. A *(substitutional) Σ-instance* of it is then any Σ-rule of the form $\sigma(\Gamma \vdash \varphi) \triangleq (\sigma[\Gamma] \vdash \sigma(\varphi))$, where σ is a Σ-substitution, in this way determining the equally-denoted unary operation on $\wp_{[\omega/(2\setminus 1)/1]}(\mathrm{Fm}^\omega_\Sigma) \times \mathrm{Fm}^1_\Sigma$. As usual, axiomatic Σ-rules are called *Σ-axioms* and are identified with their conclusions. A[n] *[axiomatic/finitary/unary] Σ-calculus* is then any set $\mathcal{C}$ of [axiomatic/finitary/unary] Σ-rules, the set of all Σ-instances of its elements being denoted by $\mathrm{SI}_\Sigma(\mathcal{C})$.

A *(propositional/sentential)* Σ*-logic* (cf., e.g., [8]) is any closure operator C over $\mathrm{Fm}^{\omega}_{\Sigma}$ that is *structural* in the sense that $\sigma[C(X)] \subseteq C(\sigma[X])$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and all $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$, that is, $\mathrm{img}\, C$ is closed under inverse Σ-substitutions, in which case we have the equivalence relation $\equiv^{\alpha}_{C} \triangleq \{\langle\phi,\psi\rangle \in \mathrm{Eq}^{\alpha}_{\Sigma} \mid C(\phi) = C(\psi)\}$, where $\alpha \in \wp_{\infty\setminus 1}(\omega)$, called the *inter-derivablity relation of* C, when $\alpha = \omega$. In this way, given any set S of [finitary] Σ-logics, $\wp(\mathrm{Fm}^{\omega}_{\Sigma}) \cap \bigcap_{C' \in S}(\mathrm{img}\, C')$ is a[n inductive] closure system over $\mathrm{Fm}^{\omega}_{\Sigma}$, closed under inverse Σ-substitutions, in which case the dual closure operator is a [finitary] Σ-logic, and so this is the complete lattice join of S. A *congruence of* C is any $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$ such that $\theta \subseteq \equiv^{\omega}_{C}$, the set of all them being denoted by $\mathrm{Con}(C)$. Then, given any $\theta, \vartheta \in \mathrm{Con}(C)$, the transitive closure $\theta \amalg \vartheta$ of $\theta \cup \vartheta$, being a congruence of $\mathfrak{Fm}^{\omega}_{\Sigma}$, is then that of C, for θ^{ω}_{C}, being an equivalence relation, is transitive. In particular, any maximal congruence of C (that exists, by Zorn Lemma, because $\mathrm{Con}(C) \ni \Delta_{\mathrm{Fm}^{\omega}_{\Sigma}}$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$ is so) is the greatest one to be denoted by $\eth(C)$, the variety $\mathrm{IV}(C)$ axiomatized by it being called the *intrinsic variety of* C (cf. [18]). Then, C is said to be *self-extensional*, whenever $\equiv^{\omega}_{C} \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, in which case $\eth(C) = \equiv^{\omega}_{C}$. Next, C is said to be *[inferentially] (in)consistent*, if $x_1 \notin (\in) C(\varnothing[\cup\{x_0\}])$ [(in which case $\equiv^{\omega}_{C} = \mathrm{Eq}^{\omega}_{\Sigma} \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, and so C is self-extensional)], the only inconsistent Σ-logic being denoted by IC. Further, a Σ-rule $\Gamma \to \Phi$ is said to be *satisfied/derivable in* C, provided $\Phi \in C(\Gamma)$, Σ-axioms satisfied in C being referred to as *theorems of* C. Next, a Σ-logic C' is said to be a *(proper) [K-]extension of* C [where $K \subseteq \infty$], whenever $(C[\restriction \wp_K(\mathrm{Fm}^{\omega}_{\Sigma})]) \subseteq (\subsetneq)(C'[\restriction \wp_K(\mathrm{Fm}^{\omega}_{\Sigma})])$, in which case C is said to be a *(proper) [K-]sublogic of* C'. In that case, C' and C are said to be *[K-]equivalent* ($C' \equiv_{[K]} C$, in symbols), provided they are [K-]extensions of one another. (In this connection, axiomatically/finitely stands for $1/\omega$, respectively.) Then, a[n axiomatic] Σ-calculus $\mathcal{C}$ is said to *axiomatize* C' *(relatively to* C*)*, if C' is the least Σ-logic (being an extension of C and) satisfying every rule in $\mathcal{C}$ [(in which case it is called an *axiomatic extension of* C)]. Further, a Σ-rule $\mathcal{R}$ is said to be *admissible in* C, provided the extension of C relatively axiomatized by $\mathcal{R}$ is axiomatically-equivalent to C. Clearly, $\mathcal{R}$ is admissible in C, whenever it is derivable in C. Then, C is said to be *structurally/deductively/inferentially complete|maximal*, whenever every Σ-rule, being admissible in C, is derivable in C. Clearly, C is structurally complete iff it has no proper axiomatically-equivalent extension. Then, as the join of the non-empty set of all Σ-logics axiomatically-equivalent to C is so, C has a unique structurally complete axiomatically-equivalent extension, called the *structural completion of* C. Furthermore, we have the finitary sublogic $C_{\exists}$ of C, defined by $C_{\exists}(X) \triangleq (\bigcup C[\wp_{\omega}(X)])$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, called the *finitarization of* C. Then, the extension of any finitary (in particular, diagonal) Σ-logic relatively axiomatized by a finitary Σ-calculus is a sublogic of its own finitarization, in which case it is equal to this, and so is finitary (in particular, the Σ-logic axiomatized by a finitary Σ-calculus is finitary; conversely, any [finitary] Σ-logic is axiomatized by the [finitary] Σ-calculus consisting of all those [finitary] Σ-rules, which are satisfied in C). Further, C is said to be *[weakly]* $\bar{\wedge}$*-conjunctive*, where $\bar{\wedge}$ is a (possibly, secondary) binary connective of Σ (tacitly fixed throughout the paper), provided $C(\phi \mathbin{\bar{\wedge}} \psi)[\supseteq] = C(\{\phi,\psi\})$, for all $\phi,\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, in which case any extension of C is so. Likewise, C is said to be *[weakly]* $\veebar$*-disjunctive*, where $\veebar$ is a (possibly, secondary) binary connective of Σ (tacitly fixed throughout the paper), provided $C(X \cup \{\phi \veebar \psi\})[\subseteq] = (C(X \cup \{\phi\}) \cap C(X \cup \{\psi\}))$, where $(X \cup \{\phi,\psi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, in which case [any extension of C is so, while the first two (viz., (2.8) with $i \in 2$) of] the following rules:

$$x_i \vdash (x_0 \veebar x_1), \tag{2.8}$$

$$(x_0 \veebar x_1) \vdash (x_1 \veebar x_0), \tag{2.9}$$

$$(x_0 \veebar x_0) \quad \vdash \quad x_0, \tag{2.10}$$

where $i \in 2$, are satisfied in C, and so in its extensions. Furthermore, C is said to *have Deduction Theorem (DT) with respect to* a (possibly, secondary) binary connective $\sqsupset$ of Σ (tacitly fixed throughout the paper), provided, for all $\phi \in X \subseteq \mathrm{Fm}_\Sigma^\omega$ and all $\psi \in C(X)$, it holds that $(\phi \sqsupset \psi) \in C(X \setminus \{\phi\})$. Then, C is said to be *weakly $\sqsupset$-implicative*, if it has DT with respect to $\sqsupset$ and satisfies the *Modus Ponens* rule:

$$\{x_0, x_0 \sqsupset x_1\} \vdash x_1, \tag{2.11}$$

in which case the following axioms:

$$x_0 \sqsupset x_0, \tag{2.12}$$

$$x_0 \sqsupset (x_1 \sqsupset x_0), \tag{2.13}$$

$$(x_0 \sqsupset (x_1 \sqsupset x_2)) \sqsupset ((x_0 \sqsupset x_1) \sqsupset (x_0 \sqsupset x_2)) \tag{2.14}$$

are satisfied in C. In general, by C^{MP} we denote the extension of C relatively axiomatized by (2.11). Likewise, C is said to be *(strongly) $\sqsupset$-implicative*, whenever it is weakly so as well as satisfies the *Peirce Law* axiom (cf. [13]):

$$(((x_0 \sqsupset x_1) \sqsupset x_0) \sqsupset x_0). \tag{2.15}$$

Next, C is said to *have Property of Weak Contraposition (PWC) with respect to* a unary $\sim \in \Sigma$ (tacitly fixed throughout the paper), provided, for all $\phi \in \mathrm{Fm}_\Sigma^\omega$ and all $\psi \in C(\phi)$, it holds that $\sim\phi \in C(\sim\psi)$. Then, C is said to be *[{axiomatically} (pre)maximally] $\sim$-paraconsistent*, provided it does not satisfy the *Ex Contradictione Quodlibet* rule:

$$\{x_0, \sim x_0\} \vdash x_1 \tag{2.16}$$

[and has no (more than one) proper $\sim$-paraconsistent {axiomatic} extension]. Likewise, C is said to be *$\sqsupset$-implicatively $\sim$-paraconsistent*, provided it does not satisfy the *Ex Contradictione Quodlibet* axiom:

$$\sim x_0 \sqsupset (x_0 \sqsupset x_1). \tag{2.17}$$

(Clearly, C is non-$\sim$-paraconsistent if[f] it is $\sqsupset$-implicatively so, whenever it satisfies (2.11) [and has DT with respect to $\sqsupset$].) In general, by $C^{[\mathrm{I}]\mathrm{NP}}$ we denote the least [$\sqsupset$-implicatively] non-$\sim$-paraconsistent extension of C, that is, the extension relatively axiomatized by (2.16) [resp. by (2.17)]. Further, C is said to be *(⟨pre⟩maximally) {axiomatically} [inferentially] $(\veebar, \sim)$-paracomplete*, whenever $(x_1 \veebar \sim x_1) \notin C(\varnothing[\cup\{x_0\}])$ (and has no ⟨more than one⟩ proper {axiomatic} [inferentially] $(\veebar, \sim)$-paracomplete extension). In general, by C^{EM} we denote the extension of C relatively axiomatized by the *Excluded Middle Law* axiom:

$$x_0 \veebar \sim x_0. \tag{2.18}$$

Finally, C is said to be *theorem-less/purely-inferential*, whenever it has no theorem, that is, $\varnothing \in (\mathrm{img}\, C)$. Likewise, C is said to be *[non-]pseudo-axiomatic*, provided $\bigcap_{k\in\omega} C(x_k) \nsubseteq [\subseteq] C(\varnothing)$ [in which case it is $(\veebar, \sim)$-paracomplete/(in)consistent iff it is inferentially so]. In general, $(\mathrm{img}\, C) \cup \{\varnothing\}$ is closed under inverse Σ-substitutions, for $\mathrm{img}\, C$ is so, in which case the dual closure operator C_{+0} is the greatest purely-inferential sublogic of C, called the *purely-inferential/theorem-less version of C*, while:

$$(C_{+0} \restriction \wp_{\infty\setminus 1}(\mathrm{Fm}_\Sigma^\omega)) = (C \restriction \wp_{\infty\setminus 1}(\mathrm{Fm}_\Sigma^\omega)), \tag{2.19}$$

in particular:

$$\equiv_C^\omega = \equiv_{C_{+0}}^\omega, \tag{2.20}$$

and so C_{+0} is self-extensional iff C is so. Likewise, $C_{-0} \triangleq ((C \upharpoonright \wp_{\infty\setminus 1}(\mathrm{Fm}^{\omega}_{\Sigma})) \cup \{\langle \varnothing, \bigcap_{k\in\omega} C(x_k)\rangle\}$ is the least non-pseudo-axiomatic extension of C called the *non-pseudo-axiomatic version of* C, in which case, by (2.19), we have:

$$(C_{+/-0})_{-/+0} = C, \tag{2.21}$$

whenever C is non-pseudo-axiomatic/purely-inferential, respectively, and so this provides an isomorphism between the posets of all non-pseudo-axiomatic and purely-inferential Σ-logics ordered by $\subseteq$.

Remark 2.4. By (2.21), the purely-inferential version of the axiomatic extension of a non-pseudo-axiomatic Σ-logic, relatively-axiomatized by an $\mathcal{A} \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, is relatively axiomatized by $\{x_0 \vdash \sigma_{+1}(\varphi) \mid \varphi \in \mathcal{A}\}$; □

Remark 2.5. Any purely-inferential inferentially consistent Σ-logic C is a proper sublogic of the unique purely-inferential inferentially inconsistent Σ-logic IC_{+0}, and so is not structurally complete, in which case IC_{+0} is the structural completion of C, for $(\mathrm{img}\,\mathrm{IC}_{+0}) = \{\mathrm{Fm}^{\omega}_{\Sigma}, \varnothing\}$, [relatively] axiomatized by $x_0 \vdash x_1$. □

A *(logical)* Σ*-matrix* (cf. [8]) is any couple of the form $\mathcal{A} = \langle \mathfrak{A}, D^{\mathcal{A}}\rangle$, where $\mathfrak{A}$ is a Σ-algebra, called the *underlying algebra of* $\mathcal{A}$, while $\lceil \mathcal{A}\rceil \triangleq A$ is called the *carrier/"underlying set" of* $\mathcal{A}$, whereas $D^{\mathcal{A}} \subseteq A$ is called the *truth predicate of* $\mathcal{A}$, elements of $A[\cap D^{\mathcal{A}}]$ being referred to as *[distinguished] values of* $\mathcal{A}$. (In general, matrices are denoted by Calligraphic letters [possibly, with indices], their underlying algebras being denoted by corresponding Fraktur letters [with same indices, if any].) This is said to be *n-valued/[in]consistent/truth(-non)-empty/truth-|false-{non-}singular*, where $n \in (\omega \setminus 1)$, provided $(|A| = n)/(D^{\mathcal{A}} \neq [=]A)/(D^{\mathcal{A}} = (\neq)\varnothing)/(|(D^{\mathcal{A}}|(A \setminus D^{\mathcal{A}}))| \in \{\notin\}2)$, respectively. Next, given any $\Sigma' \subseteq \Sigma$, $\mathcal{A}$ is said to be a *(*Σ*-)expansion of* its Σ'*-reduct* $(\mathcal{A}\upharpoonright\Sigma') \triangleq \langle \mathfrak{A}\upharpoonright\Sigma', D^{\mathcal{A}}\rangle$. (Any notation, being specified for single matrices, is supposed to be extended to classes of matrices member-wise.) Finally, $\mathcal{A}$ is said to be *finite[ly-generated]/"generated by"* a $B \subseteq A$, whenever $\mathfrak{A}$ is so.

Given any $\alpha \in \wp_{\infty[\setminus 1]}(\omega)$ [unless Σ has a nullary connective] and any class M of Σ-matrices, we have the closure operator $\mathrm{Cn}^{\alpha}_{\mathsf{M}}$ over $\mathrm{Fm}^{\alpha}_{\Sigma}$ dual to the closure system with basis $\{h^{-1}[D^{\mathcal{A}}] \mid \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})\}$, in which case:

$$\mathrm{Cn}^{\alpha}_{\mathsf{M}}(X) = (\mathrm{Fm}^{\alpha}_{\Sigma} \cap \mathrm{Cn}^{\omega}_{\mathsf{M}}(X)), \tag{2.22}$$

for all $X \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. Then, by (2.1), $\mathrm{Cn}^{\omega}_{\mathsf{M}}$ is a Σ-logic, called the *logic of* M, a Σ-logic C being said to be *[finitely-]defined by* M, provided it is [finitely-]equivalent to $\mathrm{Cn}^{\omega}_{\mathsf{M}}$, A Σ-logic is said to be *(unitary/uniform) n-valued*, where $n \in (\omega \setminus 1)$, whenever it is defined by an n-valued Σ-matrix, in which case it is finitary (cf. [8]), and so is the logic of any finite class of finite Σ-matrices.

As usual, Σ-matrices are treated as first-order model structures (viz., algebraic systems; cf. [11]) of the first-order signature $\Sigma \cup \{D\}$ with unary predicate D, any [in]finitary Σ-rule $\Gamma \vdash \phi$ being viewed as the [in]finitary equality-free basic strict Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any propositional Σ-formula ψ with the first-order atomic formula $D(\psi)$, as well as being *true/satisfied in* a class M of Σ-matrices iff it being satisfied in the logic of M.

Remark 2.6. Since any Σ-formula contains just finitely many variables, and so there is a variable not occurring in it, the logic of any class of truth-non-empty Σ-matrices is non-pseudo-axiomatic. □

Remark 2.7. Since any rule with[out] premises is [not] true in any truth-empty matrix, taking Remark 2.6 into account, given any class M of Σ-matrices, the purely-inferential/non-pseudo-axiomatic version of the logic of M is defined by $\mathsf{M} \cup / \setminus \mathsf{S}$, where S is "any non-empty class of truth-empty Σ-matrices"/"the class of all truth-empty members of M", respectively. $\Box$

Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices. A *(strict) [surjective] {matrix} homomorphism from $\mathcal{A}$ [on]to $\mathcal{B}$* is any $h \in \hom(\mathfrak{A},\mathfrak{B})$ such that $[h[A]=B$ and$]$ $D^{\mathcal{A}} \subseteq (=) h^{-1}[D^{\mathcal{B}}]$, the set of all them being denoted by $\hom^{[\mathrm{S}]}_{(\mathrm{S})}(\mathcal{A},\mathcal{B})$, in which case $\mathcal{B}/\mathcal{A}$ is said to be a *(strict) [surjective] {matrix} homomorphic image/counter-image of $\mathcal{A}/\mathcal{B}$*, respectively. Then, by (2.1), we have:

$$(\exists h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A},\mathcal{B})) \quad \Rightarrow \quad (\mathrm{Cn}^{\alpha}_{\mathcal{B}} \subseteq [=]\, \mathrm{Cn}^{\alpha}_{\mathcal{A}}), \tag{2.23}$$

$$(\exists h \in \hom^{\mathrm{S}}(\mathcal{A},\mathcal{B})) \quad \Rightarrow \quad (\mathrm{Cn}^{\alpha}_{\mathcal{A}}(\varnothing) \subseteq \mathrm{Cn}^{\alpha}_{\mathcal{B}}(\varnothing)), \tag{2.24}$$

for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$. Further, $\mathcal{A}[\neq \mathcal{B}]$ is said to be a *[proper] submatrix of $\mathcal{B}$*, whenever $\Delta_A \in \hom_{\mathrm{S}}(\mathcal{A},\mathcal{B})$, in which case we set $(\mathcal{B}\restriction A) \triangleq \mathcal{A}$. Injective/bijective strict homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ are referred to as *embeddings/isomorphisms of/from $\mathcal{A}$ into/onto $\mathcal{B}$*, in case of existence of which $\mathcal{A}$ is said to be *embeddable/isomorphic into/to $\mathcal{B}$*.

Given a Σ-matrix $\mathcal{A}$, $\chi^{\mathcal{A}} \triangleq \chi^{D^{\mathcal{A}}}_{A}$ is referred to as the *characteristic function of $\mathcal{A}$*. Then, any $\theta \in \mathrm{Con}(\mathfrak{A})$ such that $\theta \subseteq \theta^{\mathcal{A}} \triangleq (\ker \chi^{\mathcal{A}})$, in which case ν_θ is a strict surjective homomorphism from $\mathcal{A}$ onto $(\mathcal{A}/\theta) \triangleq \langle \mathfrak{A}/\theta, D^{\mathcal{A}}/\theta\rangle$, is called a *congruence of $\mathcal{A}$*, the set of all them being denoted by $\mathrm{Con}(\mathcal{A})$. Given any $\theta,\vartheta \in \mathrm{Con}(\mathcal{A})$, the transitive closure $\theta \amalg \vartheta$ of $\theta \cup \vartheta$, being a congruence of $\mathfrak{A}$, is then that of $\mathcal{A}$, for $\theta^{\mathcal{A}}$, being an equivalence relation, is transitive. In particular, any maximal congruence of $\mathcal{A}$ (that exists, by Zorn Lemma, because $\mathrm{Con}(\mathcal{A}) \ni \Delta_A$ is both non-empty and inductive, for $\mathrm{Con}(\mathfrak{A})$ is so) is the greatest one to be denoted by $\eth(\mathcal{A})$. Then, set $\Re(\mathcal{A}) \triangleq (\mathcal{A}/\eth(\mathcal{A}))$. Finally, $\mathcal{A}$ is said to be *[hereditarily] simple*, provided it has no non-diagonal congruence [and no non-simple submatrix].

Remark 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Σ-matrices and $h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A},\mathcal{B})$. Then, $\theta^{\mathcal{A}} = h^{-1}[\theta^{\mathcal{B}}]$* and $f \triangleq \{\langle \theta, h^{-1}[\theta]\rangle \mid \theta \in \mathrm{Con}(\mathfrak{B})\} : \mathrm{Con}(\mathfrak{B}) \to (\mathrm{Con}(\mathfrak{A}) \cap \wp(\ker h, A^2))$ [while $h[\theta^{\mathcal{A}}] = \theta^{\mathcal{B}}$ and $g \triangleq \{\langle \vartheta, h[\vartheta]\rangle \mid \vartheta \in (\mathrm{Con}(\mathfrak{A}) \cap \wp(\ker h, A^2))\} : (\mathrm{Con}(\mathfrak{A}) \cap \wp(\ker h, A^2)) \to \mathrm{Con}(\mathfrak{B})$, whereas $f \circ g$ and $g \circ f$ are diagonal]. Therefore,

(i) $f' \triangleq (f \restriction \mathrm{Con}(\mathcal{B})) : \mathrm{Con}(\mathcal{B}) \to (\mathrm{Con}(\mathcal{A}) \cap \wp(\ker h, A^2))$ *[while $g' \triangleq (g \restriction (\mathrm{Con}(\mathcal{A}) \cap \wp(\ker h, A^2))) : (\mathrm{Con}(\mathcal{A}) \cap \wp(\ker h, A^2)) \to \mathrm{Con}(\mathcal{B})$, whereas $f' \circ g'$ and $g' \circ f'$ are diagonal.]*

In particular (when $\theta = \Delta_B$), $(\ker h) = h^{-1}[\Delta_B] \in \mathrm{Con}(\mathcal{A})$, in which case $(\ker h) \subseteq \eth(\mathcal{A})$, and so

(ii) *h is injective, whenever $\mathcal{A}$ is simple.*

[Moreover, when $\vartheta = \eth(\mathcal{A})$ and $\theta = \eth(\mathcal{B})$, we have $h^{-1}[\theta] \subseteq \vartheta \supseteq (\ker h)$, in which case we get $\theta = h[h^{-1}[\theta]] \subseteq h[\vartheta] \subseteq \theta$, and so $\theta = h[\vartheta]$, in which case $\vartheta = h^{-1}[h[\vartheta]] = h^{-1}[\theta]$, and so

(iii) $\eth(\mathcal{B}) = h[\eth(\mathcal{A})]$ *and* $\eth(\mathcal{A}) = h^{-1}[\eth(\mathcal{B})]$. *In particular, $\mathcal{B}$ is simple, whenever $\mathcal{A}$ is so.*

In particular (when $\mathcal{B} = (\mathcal{A}/\eth(\mathcal{A}))$ and $h = \nu_{\eth(\mathcal{A})}$), we have $h[\eth(\mathcal{A})] = h[\ker h] = \Delta_B$, and so

(iv) *$\mathcal{A}/\eth(\mathcal{A})$ is simple.*] $\Box$

A Σ-matrix $\mathcal{A}$ is said to be a *[K-]model of* a (finitary) Σ-logic C {*over* $\mathfrak{A}$} [where $K \subseteq \infty$], provided C is a [K-]sublogic of the logic of $\mathcal{A}$, the class of all $\langle$simple of$\rangle$ them being denoted by $\mathrm{Mod}^{\langle *\rangle}_{[K]}(C\{,\mathfrak{A}\})$, respectively. Then, $\mathrm{Fi}_C(\mathfrak{A}) \triangleq \pi_1[\mathrm{Mod}(C,\mathfrak{A})]$, elements of which are called *C-filters of/over $\mathfrak{A}$*, is a(n inductive) closure system over A, the dual (finitary) closure operator being denoted by $\mathrm{Fg}^{\mathfrak{A}}_{C}$, in which case $\mathrm{Fi}_C(\mathfrak{Fm}^\omega_\Sigma) = (\mathrm{img}\,C)$, and so $\mathrm{Fg}^{\mathfrak{Fm}^\omega_\Sigma}_C = C$ (while, given any finitary axiomatization $\mathcal{C}$ of C and any $(X \cup \{a\}) \subseteq A$, it holds that $a \in \mathrm{Fg}^{\mathfrak{A}}_C(X)$ iff a is *derivable in* $\mathcal{C}$ *from* X *over* $\mathfrak{A}$ in the sense that there is a(n) *(abstract)* $\mathcal{C}$-*derivation of* a *from* X *over* $\mathfrak{A}$, that is, any $\bar{b} \in A^+$ such that $a \in (\mathrm{img}\,\bar{b})$ and, for each $i \in (\mathrm{dom}\,\bar{b})$, either $b_i \in X$ or there are some $(\Gamma \vdash \varphi) \in \mathcal{C}$ and some $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ such that $b_i = h(\varphi)$ and $h[\Gamma] \subseteq (\mathrm{img}(\bar{b}{\upharpoonright}i))$ — the reservation "from X"/"over $\mathfrak{A}$" is omitted, whenever $(X = \varnothing)/(\mathfrak{A} = \mathfrak{Fm}^\omega_\Sigma)$, respectively; cf. [15]). Next, $\mathcal{A}$ is said to be $\sim$*-paraconsistent/"[inferentially]* $(\veebar, \sim)$*-paracomplete"*, whenever the logic of $\mathcal{A}$ is so. Further, $\mathcal{A}$ is said to be *[weakly]* $\diamond$*-conjunctive*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, provided $(\{a, b\} \subseteq D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, that is, the logic of $\mathcal{A}$ is [weakly] $\diamond$-conjunctive. Then, $\mathcal{A}$ is said to be *[weakly]* $\diamond$*-disjunctive*, whenever $\langle \mathfrak{A}, A \setminus D^{\mathcal{A}} \rangle$, in which case [resp., that is] the logic of $\mathcal{A}$ is [weakly] $\diamond$-disjunctive, and so is the logic of any class of [weakly] $\diamond$-disjunctive Σ-matrices. Likewise, $\mathcal{A}$ is said to be $\diamond$*-implicative*, whenever $((a \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$, in which case it is $\uplus_\diamond$-disjunctive, while the logic of $\mathcal{A}$ is $\diamond$-implicative, for both (2.11) and $(2.15) = ((x_0 \sqsupset x_1) \uplus_\sqsupset x_0)$ are true in any $\sqsupset$-implicative (and so $\uplus_\sqsupset$-disjunctive) Σ-matrix, while DT is immediate, and so is the logic of any class of $\diamond$-implicative Σ-matrices. Finally, given any (possibly secondary) unary connective $\wr$ of Σ, put $(x_0 \diamond^\wr x_1) \triangleq \wr(\wr x_0 \diamond \wr x_1)$ and $(x_0 \sqsupset^\wr_\diamond x_1) \triangleq (\wr x_0 \diamond x_1)$. Then, $\mathcal{A}$ is said to be *[weakly] (classically)* $\wr$*-negative*, provided, for all $a \in A$, $(a \in D^{\mathcal{A}})[\Leftarrow] \Leftrightarrow (\wr^{\mathfrak{A}} a \notin D^{\mathcal{A}})$, in which case it is [truth-non-empty], and so consistent.

Remark 2.9. Let $\diamond$ and $\wr$ be as above. Then, the following hold:

(i) any (weakly) $\wr$-negative Σ-matrix $\mathcal{A}$:
  a) is [weakly] $\diamond$-disjunctive/-conjunctive iff it is [weakly] $\diamond^\wr$-conjunctive/-disjunctive, respectively;
  b) defines a logic having PWC with respect to $\wr \in \Sigma$;
  c) is $\sqsupset^\wr_\diamond$-implicative, whenever it is $\diamond$-disjunctive;
  d) is not $\wr$-paraconsistent(/$(\diamond, \wr)$-paracomplete), whenever $\wr \in \Sigma$(/ while $\mathcal{A}$ is weakly $\diamond$-disjunctive).

(ii) given any two Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ and any $h \in \hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$, $\mathcal{A}$ is (weakly) $\wr$-negative|$\diamond$-conjunctive/-disjunctive/-implicative if[f] $\mathcal{B}$ is so;

(iii) the direct product of any tuple of Σ-matrices is not $\wr$-paraconsistent, where $\wr \in \Sigma$, whenever the tuple image contains a non-$\wr$-paraconsistent consistent Σ-matrix. □

Remark 2.10. Given a Σ-logic C, by its structurality, for any $T \in (\mathrm{img}\,C)$, $\langle \mathfrak{Fm}^\omega_\Sigma, T \rangle \in \mathrm{Mod}(C)$. Then, given any basis $\mathcal{B}$ of $\mathrm{img}\,C$, any Σ-rule $\Gamma \vdash \varphi$ not satisfied in C, in which case there is some $T \in \mathcal{B}$ such that $\Gamma \subseteq T \not\ni \varphi$, is not true in $\langle \mathfrak{Fm}^\omega_\Sigma, T \rangle$ under the diagonal Σ-substitution, and so C is defined by $\{\langle \mathfrak{Fm}^\omega_\Sigma, T \rangle \mid T \in \mathcal{B}\}$. □

Given a set I and an I-tuple $\overline{\mathcal{A}}$ of Σ-matrices, [any submatrix $\mathcal{B}$ of] the Σ-matrix $(\prod_{i \in I} \mathcal{A}_i) \triangleq \langle \prod_{i \in I} \mathfrak{A}_i, \prod_{i \in I} D^{\mathcal{A}_i} \rangle$ is called the [a] *[sub]direct product of* $\overline{\mathcal{A}}$ [whenever, for each $i \in I$, $\pi_i[B] = A_i$]. As usual, if $(\mathrm{img}\,\overline{\mathcal{A}}) \subseteq \{\mathcal{A}\}$ (and $I = 2$), where $\mathcal{A}$ is a Σ-matrix, $\mathcal{A}^I \triangleq (\prod_{i \in I} \mathcal{A}_i)$ [resp., $\mathcal{B}$] is called the [a] *[sub]direct I-power (square) of* $\mathcal{A}$.

Given a class M of Σ-matrices, the class of all "strict surjective homomorphic [counter-]images"/"(consistent) submatrices" of members of M is denoted, respectively, by $(\mathbf{H}^{[-1]}/\mathbf{S}_{(*)})(\mathsf{M})$. Likewise, the class of all [sub]direct products of tuples (of cardinality $\in K \subseteq \infty$) constituted by members of M is denoted by $\mathbf{P}^{[\mathrm{SD}]}_{(K)}(\mathsf{M})$.

Lemma 2.11. *Let* M *be a class of* Σ*-matrices. Then,* $\mathbf{H}(\mathbf{H}^{-1}(\mathsf{M})) \subseteq \mathbf{H}^{-1}(\mathbf{H}(\mathsf{M}))$.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be Σ-matrices, $\mathcal{C} \in \mathsf{M}$ and $(h|g) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \mathcal{C}|\mathcal{A})$. Then, by Remark 2.8(i), $(\ker(h|g)) \in \mathrm{Con}(\mathcal{B})$, in which case $(\ker(h|g)) \subseteq \theta \triangleq \eth(\mathcal{B}) \in \mathrm{Con}(\mathcal{B})$, and so, by the Homomorphism Theorem, $(\nu_\theta \circ (h|g)^{-1}) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{C}|\mathcal{A}, \mathcal{B}/\theta)$. □

Lemma 2.12 (Finite Subdirect Product Lemma; cf. Lemma 2.7 of [25])**.** *Let* M *be a finite class of finite* Σ*-matrices and* $\mathcal{A}$ *a finitely-generated (in particular, finite) model of the logic of* M*. Then,* $\mathcal{A} \in \mathbf{H}^{-1}(\mathbf{H}(\mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathsf{M}))))$.

Theorem 2.13 (cf. Theorem 2.8 of [25])**.** *Let* K *and* M *be classes of* Σ*-matrices,* C *the logic of* M *and* C' *an extension of* C*. Suppose [both* M *and all members of it are finite and]* $\mathbf{P}^{\mathrm{SD}}_{[\omega]}(\mathbf{S}_*(\mathsf{M})) \subseteq \mathsf{K}$ *(in particular,* $\mathbf{S}(\mathbf{P}_{[\omega]}(\mathsf{M})) \subseteq \mathsf{K}$ *{in particular,* $\mathsf{K} \supseteq \mathsf{M}$ *is closed under both* $\mathbf{S}$ *and* $\mathbf{P}_{[\omega]}\langle$ *in particular,* $\mathsf{K} = \mathrm{Mod}(C)\rangle\}$*). Then,* C' *is [finitely-]defined by* $\mathrm{Mod}(C') \cap \mathsf{K}$*, and so by* $\mathrm{Mod}(C')$.

Corollary 2.14 (cf. Corollary 2.9 of [25])**.** *Let* M *be a class of* Σ*-matrices and* $\mathcal{A}$ *an axiomatic* Σ*-calculus. Then, the axiomatic extension of the logic of* M *relatively axiomatized by* $\mathcal{A}$ *is defined by* $\mathbf{S}_*(\mathsf{M}) \cap \mathrm{Mod}(\mathcal{A})$.

Given any Σ-logic C and any $\Sigma' \subseteq \Sigma$, in which case $\mathrm{Fm}^{\alpha}_{\Sigma'} \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$ and $\hom(\mathfrak{Fm}^{\alpha}_{\Sigma'}, \mathfrak{Fm}^{\alpha}_{\Sigma'}) = \{h \restriction \mathrm{Fm}^{\alpha}_{\Sigma'} \mid h \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{Fm}^{\alpha}_{\Sigma}), h[\mathrm{Fm}^{\alpha}_{\Sigma'}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma'}\}$, for all $\alpha \in \wp_{\infty\setminus 1}(\omega)$, we have the Σ'-logic C', defined by $C'(X) \triangleq (\mathrm{Fm}^{\omega}_{\Sigma'} \cap C(X))$, for all $X \subseteq \mathrm{Fm}^{\omega}_{\Sigma'}$, called the Σ'*-fragment of* C, in which case C is said to be a *(Σ-)expansion of* C', while, given any class M of Σ-matrices, C' is defined by $\mathsf{M}\restriction\Sigma'$, whenever C is defined by M, whereas $\equiv^{\omega}_{C'} = (\equiv^{\omega}_{C} \cap \mathrm{Eq}^{\omega}_{\Sigma'})$, and so C' is self-extensional, whenever C is so.

3. Preliminary key adnanced generic issues

3.1. False-singular consistent weakly conjunctive matrices.

Lemma 3.1. *Let* $\mathcal{A}$ *be a false-singular weakly* $\bar{\wedge}$*-conjunctive* Σ*-matrix,* $f \in (A \setminus D^{\mathcal{A}})$*,* I *a finite set,* $\overline{\mathcal{C}}$ *an* I*-tuple constituted by consistent submatrices of* $\mathcal{A}$ *and* $\mathcal{B}$ *a subdirect product of* $\overline{\mathcal{C}}$*. Then,* $(I \times \{f\}) \in B$.

Proof. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $a \in B$ including $(J \times \{f\})$. First, when $J = \varnothing$, take any $a \in C \neq \varnothing$, in which case $(J \times \{f\}) = \varnothing \subseteq a$. Now, assume $J \neq \varnothing$. Take any $j \in J \subseteq I$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, and so, as $\mathcal{C}_j$ is a consistent submatrix of the false-singular Σ-matrix $\mathcal{A}$, we have $f \in C_j = \pi_j[B]$. Hence, there is some $b \in B$ such that $\pi_j(b) = f$, while, by induction hypothesis, there is some $a \in B$ including $(K \times \{f\})$. Therefore, since $J = (K \cup \{j\})$, while $\mathcal{A}$ is both weakly $\bar{\wedge}$-conjunctive and false-singular, we have $B \ni c \triangleq (a \mathbin{\bar{\wedge}^{\mathfrak{B}}} b) \supseteq (J \times \{f\})$. Thus, when $J = I$, we eventually get $B \ni (I \times \{f\})$, as required. □

3.2. Congruence and equality determinants versus matrix simplicity and intrinsic varieties.

A *[binary] relational* Σ*-scheme* is any Σ-calculus [of the form] $\varepsilon \subseteq (\wp(\mathrm{Fm}^{[2\cap]\omega}_{\Sigma}) \times \mathrm{Fm}^{[2\cap]\omega}_{\Sigma})$, in which case, given any Σ-matrix $\mathcal{A}$, we set $\theta^{\mathcal{A}}_{\varepsilon} \triangleq \{\langle a, b\rangle \in A^2 \mid \mathcal{A} \models (\forall_{([2\cap]\omega)\setminus 2} \bigwedge \varepsilon)[x_0/a, x_1/b]\} \subseteq A^2$. Given a one more Σ-matrix $\mathcal{B}$ and any $h \in \hom^{(\mathrm{S})}_{\{\mathrm{S}\}}(\mathcal{A}, \mathcal{B})$, being strict, unless ε is axiomatic, we have:

$$h^{-1}[\theta^{\mathcal{B}}_{\varepsilon}]\{\subseteq\}(\supseteq)[\supseteq]\theta^{\mathcal{A}}_{\varepsilon}. \tag{3.1}$$

A *[unary] unitary relational* Σ*-scheme* is any $\Upsilon \subseteq \mathrm{Fm}^{[1\cap]\omega}_{\Sigma}$, in which case we have the unary [binary] relational Σ-scheme $\varepsilon_\Upsilon \triangleq \{(\upsilon[x_0/x_i]) \vdash (\upsilon[x_0/x_{1-i}]) \mid i \in 2, \upsilon \in \sigma_{1:+1}[\Upsilon]\}$ such that $\theta^{\mathcal{A}}_{\varepsilon_\Upsilon}$, where $\mathcal{A}$ is any Σ-matrix, is an equivalence relation on A.

A *[binary] congruence/equality determinant for* a class of Σ-matrices M is any [binary] relational Σ-scheme ε such that, for each $\mathcal{A} \in \mathsf{M}$, $\theta^{\mathcal{A}}_{\varepsilon} \in \mathrm{Con}(\mathcal{A}) / = \Delta_A$, respectively, that includes a finite one, whenever both M and all members of it are finite.

Then, according to [22]/[21], a *[unary] unitary congruence/equality determinant for* a class of Σ-matrices M is any [unary] unitary relational Σ-scheme Υ such that ε_Υ is a/an congruence/equality determinant for M that includes a finite one, whenever both M and all members of it are finite. (It is unary unitary equality determinants that are equality determinants in the sense of [21].)

Lemma 3.2. *Let $\mathcal{A}$ be a Σ-matrix, $\theta \in \mathrm{Con}(\mathcal{A})$ and ε a relational Σ-scheme. Then, $\theta \subseteq \theta^{\mathcal{A}}_{\varepsilon}$, whenever $\Delta_A \subseteq \theta^{\mathcal{A}}_{\varepsilon}$. In particular, $\eth(\mathcal{A}) = \theta^{\mathcal{A}}_{\varepsilon}$ iff ε is a congruence determinant for $\mathcal{A}$, in which case $\mathcal{A}$ is simple iff ε is an equality determinant for it.*

Proof. Let $\mathcal{B} \triangleq (\mathcal{A}/\theta)$, in which case $h \triangleq \nu_\theta \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$. Consider any $\langle a, b\rangle \in \theta$, in which case $h(a) = h(b)$. Therefore, if $\Delta_A \subseteq \theta^{\mathcal{A}}_{\varepsilon}$, then we have $\langle a, a\rangle \in \theta^{\mathcal{A}}_{\varepsilon}$, in which case, by (3.1), we get $\langle h(a), h(b)\rangle = \langle h(a), h(a)\rangle \in \theta^{\mathcal{B}}_{\varepsilon}$, and so we eventually get $\langle a, b\rangle \in \theta^{\mathcal{A}}_{\varepsilon}$, as required. □

Corollary 3.3 (cf., e.g., [22])**.** $\mathrm{Fm}^{\omega}_{\Sigma}$ *is a unitary congruence determinant for every Σ-matrix $\mathcal{A}$.*

Proof. Since $\Delta_A \subseteq \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}$, by Lemma 3.2, we have $\eth(\mathcal{A}) \subseteq \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}$. Conversely, consider any $\bar{a} \in (A^2 \setminus \eth(\mathcal{A}))$. Then, as $\bar{a} \in (\mathrm{Cg}^{\mathfrak{A}}(\bar{a}) \setminus \eth(\mathcal{A}))$, $\theta \triangleq \mathrm{Cg}^{\mathfrak{A}}(\bar{a}) \nsubseteq \eth(\mathcal{A})$, in which case $\mathrm{Con}(\mathfrak{A}) \ni \theta \notin \mathrm{Con}(\mathcal{A})$, and so $\theta \nsubseteq \theta^{\mathcal{A}}$. Let

$$\vartheta \triangleq \{\langle \varphi^{\mathfrak{A}}[x_0/a_i; x_{j+1}/c_j]_{j\in(n-1)}, \varphi^{\mathfrak{A}}[x_0/a_{1-i}; x_{j+1}/c_j]_{j\in(n-1)}\rangle \mid i \in 2, n \in (\omega \setminus 1), \varphi \in \mathrm{Fm}^{n}_{\Sigma}, \bar{c} \in A^{n-1}\}.$$

Then, by Mal'cev's Principal Congruence Lemma [10], θ is the transitive closure of ϑ. Hence, $\theta^{\mathcal{A}}$, being transitive, does not include ϑ, in which case there are some $i \in 2$, some $n \in (\omega \setminus 1)$, some $\varphi \in \mathrm{Fm}^{n}_{\Sigma}$ and some $\bar{c} \in A^{n-1}$ such that $\varphi^{\mathfrak{A}}[x_0/a_i; x_{j+1}/c_j]_{j\in(n-1)} \in D^{\mathcal{A}} \not\ni \varphi^{\mathfrak{A}}[x_0/a_{1-i}; x_{j+1}/c_j]_{j\in(n-1)}$, in which case $\bar{a} \notin \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}$, and so $\theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}} = \eth(\mathcal{A}) \in \mathrm{Con}(\mathcal{A})$, as required. □

Corollary 3.4. *Let C be a Σ-logic, $\theta \in \mathrm{Con}(C)$, $\mathcal{A} \in \mathrm{Mod}(C)$ and $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, $h[\theta] \subseteq \eth(\mathcal{A})$.*

Proof. Consider any $\langle \phi, \psi\rangle \in \theta$, any $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $g(x_{0/1}) = h(\phi/\psi)$ and any $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$. Then, $V \triangleq (\mathrm{Var}(\sigma_{1:+1}(\varphi)) \setminus \{x_0\}) \in \wp_\omega(V_\omega)$. Let $n \triangleq |V| \in \omega$ and $\bar{v}$ any enumeration of V. Likewise, $U \triangleq (\bigcup \mathrm{Var}[\{\phi, \psi\}]) \in \wp_\omega(V_\omega)$, in which case $V_\omega \setminus U$ is infinite, and so there is an injective $\bar{u} \in (V_\omega \setminus U)^n$. Then, by the reflexivity of $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, we have $\xi \triangleq (\sigma_{1:+1}(\varphi)[x_0/\phi; v_i/u_i]_{i\in n})\ \theta\ \eta \triangleq (\sigma_{1:+1}(\varphi)[x_0/\psi; v_i/u_i]_{i\in n})$. Let $f \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h{\restriction}(V_\omega \setminus (\mathrm{img}\,\bar{u}))) \cup [u_i/g(v_i)]_{i\in n}$. Then, as $\mathcal{A} \in \mathrm{Mod}(C)$ and $\theta \subseteq \equiv^{\omega}_{C}$, we get $g(\sigma_{1:+1}(\varphi)) = f(\xi)\ \theta^{\mathcal{A}}\ f(\eta) = g(\sigma_{1:+1}(\varphi)[x_0/x_1])$. In this way, $h(\phi)\ \theta^{\mathcal{A}}_{\varepsilon_{\mathrm{Fm}^{\omega}_{\Sigma}}}\ h(\psi)$, and so Corollary 3.3 completes the argument. □

As a particular case of Corollary 3.4, we first have:

Corollary 3.5. *Let C be a Σ-logic. Then, $\pi_0[\mathrm{Mod}^*(C)] \subseteq \mathrm{IV}(C)$.*

Corollary 3.6. *Let C be a Σ-logic. Then, $\eth(C)$ is fully invariant. In particular, $\eth(C) = \theta^{\omega}_{\mathrm{IV}(C)}$.*

Proof. Consider any $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ and any $T \in (\mathrm{img}\,C)$, in which case, by the structurality of C, $\mathcal{A}_T \triangleq \langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle \in \mathrm{Mod}(C)$, and so, by Corollary 3.4, $\sigma[\eth(C)] \subseteq \eth(\mathcal{A}_T)$. Thus, $\sigma[\eth(C)] \subseteq \theta \triangleq (\mathrm{Eq}^{\omega}_{\Sigma} \cap \bigcap\{\eth(\mathcal{A}_T) \mid T \in (\mathrm{img}\,C)\}) \subseteq$

$(\mathrm{Eq}^{\omega}_{\Sigma} \cap \bigcap\{\theta^{\mathcal{A}_T} \mid T \in (\mathrm{img}\, C)\}) = \equiv^{\omega}_{C}$. Moreover, for each $T \in (\mathrm{img}\, C)$, $\partial(\mathcal{A}_T) \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, in which case $\theta \in \mathrm{Con}(\mathfrak{Fm}^{\omega}_{\Sigma})$, and so $\sigma[\partial(C)] \subseteq \theta \subseteq \partial(C)$. □

Lemma 3.7. *Let* M *be a class of* Σ*-matrices,* $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ *and* C *the logic of* M*. Then,* $\theta^{\omega}_{\mathsf{K}} \subseteq \equiv^{\omega}_{C}$*, in which case* $\theta^{\omega}_{\mathsf{K}} \subseteq \partial(C)$*, and so* $\mathrm{IV}(C) \subseteq \mathbf{V}(\mathsf{K})$*.*

Proof. Then, for any $\langle \phi, \psi \rangle \in \theta^{\omega}_{\mathsf{K}}$, each $\mathcal{A} \in \mathsf{M}$ and all $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$, $\mathfrak{A} \in \mathsf{K}$, in which case $\langle h(\phi), h(\psi) \rangle \in \Delta_A \subseteq \theta^{\mathcal{A}}$, and so $\phi \equiv^{\omega}_{C} \psi$, as required. □

By Corollary 3.5 and Lemma 3.7, we immediately have:

Corollary 3.8. *Let* M *be a class of* Σ*-matrices,* $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ *and* C *the logic of* M*. Then,* $\pi_0[\mathrm{Mod}^*(C)] \subseteq \mathbf{V}(\mathsf{K})$*.*

Likewise, by Corollary 3.5 and Lemma 3.7, we also have:

Theorem 3.9. *Let* M *be a class of simple* Σ*-matrices,* $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ *and* C *the logic of* M*. Then,* $\mathrm{IV}(C) = \mathbf{V}(\mathsf{K})$*.*

Lemma 3.10. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* Σ*-matrices,* ε *a/an congruence/equality determinant for* $\mathcal{B}$ *and* h *a/an "strict homomorphism"/embedding from/of* $\mathcal{A}$ *to/into* $\mathcal{B}$*. Suppose either* ε *is binary or* $h[A] = B$*. Then,* ε *is a/an congruence/equality determinant for* $\mathcal{A}$*.*

Proof. In that case, by (3.1), we have $\theta^{\mathcal{A}}_{\varepsilon} = h^{-1}[\theta^{\mathcal{B}}_{\varepsilon}]$. In this way, Remark 2.8(i)/"the injectivity of h" completes the argument. □

Theorem 3.11. *Let* $\mathcal{A}$ *be a* Σ*-matrix. Then, the following are equivalent:*

(i) $\mathcal{A}$ *is hereditarily simple;*
(ii) $\mathcal{A}$ *has a binary equality determinant;*
(iii) $\mathcal{A}$ *has a unary binary equality determinant.*

Proof. First, (ii) is a particular case of (iii), (ii)⇒(i) being by Lemmas 3.2 and 3.10.

Finally, assume (i) holds. Let $\varepsilon \triangleq \{\phi_i \vdash \phi_{1-i} \mid i \in 2, \bar{\phi} \in (\mathrm{Fm}^2_{\Sigma})^2, (\phi_0[x_1/x_0]) = (\phi_1[x_1/x_0])\}$. Then, $\Delta_A \subseteq \theta^{\mathcal{A}}_{\varepsilon}$. Conversely, consider any distinct $\bar{a} \in (A^2 \setminus \Delta_A)$. Let $\mathcal{B}$ be the submatrix of $\mathcal{A}$ generated by $\mathrm{img}\, \bar{a}$. Then, it is simple, by (i). Therefore, $\theta \triangleq \mathrm{Cg}^{\mathfrak{B}}(\bar{a}) \nsubseteq \theta^{\mathcal{B}}$, for $\theta \ni \bar{a} \notin \Delta_B$ is a non-diagonal congruence of $\mathfrak{B}$. Let $\vartheta \triangleq \{\langle \varphi^{\mathfrak{B}}[x_0/a_j; x_{k+1}/c_k]_{k \in (n-1)}, \varphi^{\mathfrak{B}}[x_0/a_{1-j}; x_{k+1}/c_k]_{k \in (n-1)} \rangle \mid j \in 2, n \in (\omega \setminus 1), \varphi \in \mathrm{Fm}^n_{\Sigma}, \bar{c} \in B^{n-1}\}$. Then, by Mal'cev's Principal Congruence Lemma [10], θ is the transitive closure of ϑ. Hence, $\theta^{\mathcal{B}}$, being transitive, does not include ϑ, in which case there are some $j \in 2$, some $n \in (\omega \setminus 1)$, some $\varphi \in \mathrm{Fm}^n_{\Sigma}$ and some $\bar{c} \in B^{n-1}$ such that $\langle \varphi^{\mathfrak{B}}[x_0/a_j; x_{k+1}/c_k]_{k \in (n-1)}, \varphi^{\mathfrak{B}}[x_0/a_{1-j}; x_{k+1}/c_k]_{k \in (n-1)} \rangle \notin \theta^{\mathcal{B}}$, in which case there is some $i \in 2$ such that $\varphi^{\mathfrak{B}}[x_0/a_i; x_{k+1}/c_k]_{k \in (n-1)} \in D^{\mathcal{B}} \not\ni \varphi^{\mathfrak{B}}[x_0/a_{1-i}; x_{k+1}/c_k]_{k \in (n-1)}$, while, as $\mathfrak{B}$ is generated by $\mathrm{img}\, \bar{a}$, there is some $\bar{\psi} \in (\mathrm{Fm}^2_{\Sigma})^{n-1}$ such that $c_k = \psi^{\mathfrak{B}}[x_l/a_l]_{l \in 2}$, for all $k \in (n-1)$, and so $\phi^{\mathfrak{B}}_i[x_l/a_l]_{l \in 2} \in D^{\mathcal{B}} \not\ni \phi^{\mathfrak{B}}_{1-i}[x_l/a_l]_{l \in 2}$, where, for each $m \in 2$, $\phi_m \triangleq (\varphi[x_0/x_m; x_{k+1}/\psi_k]_{k \in (n-1)} \in \mathrm{Fm}^2_{\Sigma}$. Moreover, $(\phi_0[x_1/x_0]) = (\varphi[x_{k+1}/(\psi_k[x_0/x_1])]_{k \in (n-1)} = (\phi_1[x_1/x_0])$, in which case $(\phi_i \vdash \phi_{1-i}) \in \varepsilon$, and so $\bar{a} \notin \theta^{\mathcal{B}}_{\varepsilon} = (\theta^{\mathcal{A}}_{\varepsilon} \cap B^2)$, in view of (3.1) with $h = \Delta_B$ as well as $\mathcal{A}$ and $\mathcal{B}$ instead of one another. Thus, $\bar{a} \notin \theta^{\mathcal{A}}_{\varepsilon}$, for $\bar{a} \in B^2$, in which case ε is a unary binary equality determinant for $\mathcal{A}$, and so (iii) holds. □

Lemma 3.12. *Let* $\mathcal{A}$ *be a* Σ*-matrix with unary unitary equality determinant* Υ*,* $\mathcal{B}$ *a submatrix of* $\mathcal{A}$ *and* $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$*. Then,* h *is diagonal.*

Proof. Consider any $a \in B$. Then, for any $\upsilon \in \Upsilon$, we have $(\upsilon^{\mathfrak{A}}(a) = \upsilon^{\mathfrak{B}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(h(a)) = h(\upsilon^{\mathfrak{B}}(a)) \in D^{\mathcal{A}})$, so we get $h(a) = a$, as required. □

Lemma 3.13. *Any axiomatic binary equality determinant ε for a class M of Σ-matrices is so for $\mathbf{P}(\mathsf{M})$.*

Proof. In that case, members of M are models of the infinitary universal strict Horn theory $\varepsilon[x_1/x_0] \cup \{(\bigwedge \varepsilon) \to (x_0 \approx x_1)\}$ with equality, and so are well-known to be those of $\mathbf{P}(\mathsf{M})$, as required. □

3.3. **Disjunctivity.** Fix any set A, any closure operator C over A and any $\delta : A^2 \to A$, in which case we set $\delta(X,Y) \triangleq \delta[X \times Y]$, for all $X, Y \subseteq A$.

Then, any $X \subseteq A$ is said to be δ-*disjunctive*, provided, for all $a, b \in A$, $(\delta(a,b) \in X) \Leftrightarrow ((\{a,b\} \cap X) \neq \varnothing)$, in which case, for all $Y, Z \subseteq A$, $(\delta(Y,Z) \subseteq X) \Leftrightarrow ((Y \subseteq X)|(Z \subseteq X))$.

Next, C is said to be δ-*disjunctive*, provided, for all $a, b \in A$ and every $X \subseteq A$, it holds that

$$C(X \cup \{\delta(a,b)\}) = (C(X \cup \{a\}) \cap C(X \cup \{b\})), \tag{3.2}$$

in which case the following clearly hold, by (3.2) with $X = \varnothing$:

$$\begin{aligned} \delta(a,b) &\in C(a), && (3.3)\\ \delta(a,b) &\in C(b), && (3.4)\\ a &\in C(\delta(a,a)), && (3.5)\\ \delta(b,a) &\in C(\delta(a,b)), && (3.6) \end{aligned}$$

and so, by (3.2), (3.3) and (3.4), does:

$$\delta(C(X \cup \{b\}), a) \subseteq C(X \cup \{\delta(b,a)\}). \tag{3.7}$$

Conversely, we have:

Lemma 3.14. *Suppose either* (3.3) *or* (3.4) *as well as both* (3.5), (3.6) *and* (3.7) *hold. Then, C is δ-disjunctive.*

Proof. In that case, by (3.6), both (3.3) and (3.4) hold, and so does the inclusion from left to right in (3.2). Conversely, consider any $c \in (C(X \cup \{b\}) \cap C(X \cup \{a\}))$, where $(X \cup \{a,b\}) \subseteq A$. Then, by (3.6) and (3.7), we have $\delta(b,c) \in C(X \cup \{\delta(a,b)\})$. Likewise, by (3.5) and (3.7), we have $c \in C(X \cup \{\delta(b,c)\})$. Therefore, we eventually get $c \in C(X \cup \{\delta(a,b)\})$, as required. □

Likewise, C is said to be δ-*multiplicative*, provided

$$\delta(C(X), a) \subseteq C(\delta(X,a)), \tag{3.8}$$

for all $X \subseteq A$ and all $a \in A$.

Lemma 3.15. *Let $\mathcal{B}$ be a basis of $\operatorname{img} C$. Suppose every element of $\mathcal{B}$ is δ-disjunctive. Then, C is δ-multiplicative, while* (3.3), (3.4), (3.5) *and* (3.6) *hold.*

Proof. Consider any $X \subseteq A$, any $a \in A$ and any $Z \in \mathcal{B} \subseteq (\operatorname{img} C)$. Then, $Z = C(Z)$ is δ-disjunctive, in which case we have $(\delta(X,a) \subseteq Z) \Rightarrow ((X \subseteq Z)|(a \in Z)) \Rightarrow ((C(X) \subseteq Z)|(a \in Z)) \Rightarrow (\delta(C(X),a) \subseteq Z)$, and so (3.8) does hold. Moreover, $\delta(a,b) \in Z$ iff $\{a,b\} = \{b,a\}$ is not disjoint with Z, so (3.3), (3.4) and (3.6) hold. Finally, $\delta(a,a) \in Z$ iff $\{a,a\} = \{a\}$ is not disjoint with Z, so (3.5) holds too, as required. □

Corollary 3.16. *Suppose C is finitary. Then, the following are equivalent:*

(i) *C is δ-disjunctive;*
(ii) *$\operatorname{img} C$ has a basis consisting of δ-disjunctive sets;*
(iii) *C is δ-multiplicative, while either* (3.3) *or* (3.4) *as well as both* (3.5) *and* (3.6) *hold.*

Proof. First, assume (i) holds. Then, by Remark 2.1, $\mathcal{B} \triangleq \mathrm{MI}(\mathrm{img}\, C)$ is a basis of $\mathrm{img}\, C$. Consider any $a, b \in A$ and any $X \in \mathcal{B}$. Then, in case $(a/b) \in X$, by (3.3)/(3.4), $\delta(a,b) \in X$. Conversely, assume $\delta(a,b) \in X$. Then, by (3.2), $X = C(X) = C(X \cup \{\delta(a,b)\}) = (C(X \cup \{a\}) \cap C(X \cup \{b\}))$, in which case X, being meet-irreducible in $(\mathrm{img}\, C) \supseteq \{C(X \cup \{a\}), C(X \cup \{b\})\}$, is equal to either $C(X \cup \{a\}) \ni a$ or $C(X \cup \{b\}) \ni b$, and so X is δ-disjunctive. Thus, (ii) holds.

Next, (ii)$\Rightarrow$(iii) is by Lemma 3.15.

Finally, assume (iii) holds, and so does (3.4), in which case, by (3.8), (3.7) holds too. Then, Lemma 3.14 completes the argument of (i). □

3.3.1. *Disjunctive logics versus disjunctive matrices.*

Corollary 3.17. *Let C be a finitary $\veebar$-disjunctive Σ-logic and $\mathfrak{A}$ a Σ-algebra. Then, $\mathrm{Fg}^{\mathfrak{A}}_{C}$ is $\veebar^{\mathfrak{A}}$-disjunctive.*

Proof. By (2.8) with $i = 0$, (2.9) and (2.10), (3.3), (3.5) and (3.6) with $\mathrm{Fg}^{\mathfrak{A}}_{C}$ instead of C hold. Let $\mathcal{C}$ be a finitary Σ-calculus axiomatizing C. Consider any $X \subseteq A$, any $a, b \in A$ and any $c \in \mathrm{Fg}^{\mathfrak{A}}_{C}(X \cup \{b\})$. Then, there is some $\mathcal{C}$-derivation $\bar{d}$ of c from $X \cup \{b\}$. By complete induction on any $j \in (\mathrm{dom}\, \bar{d}) \in (\omega \setminus 1)$, let us prove that $(d_j \veebar^{\mathfrak{A}} a) \in F \triangleq \mathrm{Fg}^{\mathfrak{A}}_{C}(X \cup \{(b \veebar^{\mathfrak{A}} a)\})$. For consider the following complementary cases:

- $d_j \in (X \cup \{b\})$.
  Then, by (2.8) with $i = 0$, $(d_j \veebar^{\mathfrak{A}} a) \in ((X \cup \{b\}) \veebar^{\mathfrak{A}} a) = ((X \veebar^{\mathfrak{A}} a) \cup \{(b \veebar^{\mathfrak{A}} a)\}) \subseteq F$.
- $d_j \notin (X \cup \{b\})$.
  Then, there are some $(\Gamma \vdash \varphi) \in \mathcal{C}$ and some $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $d_j = h(\varphi)$ and $h[\Gamma] \subseteq (\mathrm{img}(\bar{d} \restriction j))$. Moreover, by the structurality of C and Corollary 3.16(i)$\Rightarrow$(iii), $(\sigma_{+1}(\varphi) \veebar x_0) \in C((\sigma_{+1}[\Gamma] \veebar x_0)$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $[x_0/a; x_{k+1}/h(x_k)]_{k \in \omega}$. Then, by induction hypothesis, we have $g[\sigma_{+1}[\Gamma] \veebar x_0] = (h[\Gamma] \veebar^{\mathfrak{A}} a) \subseteq F \in \mathrm{Fi}_C(\mathfrak{A})$. Hence, since $\langle \mathfrak{A}, F \rangle \in \mathrm{Mod}(C)$, we get $(d_j \veebar^{\mathfrak{A}} a) = (h(\varphi) \veebar^{\mathfrak{A}} a) = g(\sigma_{+1}(\varphi) \veebar x_0) \in F$.

Thus, as $c \in (\mathrm{img}\, \bar{d})$, we conclude that $(c \veebar^{\mathfrak{A}} a) \in F$, in which case (3.7) holds, and so Lemma 3.14 completes the argument. □

By Remark 2.10 and Corollary 3.16(i)$\Rightarrow$(ii), we immediately have:

Theorem 3.18. *A [finitary] Σ-logic is $\veebar$-disjunctive if[f] it is defined by a class of $\veebar$-disjunctive Σ-matrices.*

3.3.1.1. Disjunctive models of finitely-valued disjunctive logics.

Lemma 3.19. *Let M be a class of weakly $\veebar$-disjunctive Σ-matrices, I a finite set, $\overline{\mathcal{C}} \in \mathsf{M}^I$, and $\mathcal{D}$ a consistent $\veebar$-disjunctive submatrix of $\prod \overline{\mathcal{C}}$. Then, there is some $i \in I$ such that $(\pi_i \restriction D) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$.*

Proof. By contradiction. For suppose that, for every $i \in I$, $(\pi_i \restriction D) \notin \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{C}_i)$, in which case $D^{\mathcal{D}} \subsetneq (\pi_i \restriction D)^{-1}[D^{\mathcal{C}_i}] = (D \cap \pi_i^{-1}[D^{\mathcal{C}_i}])$, for $(\pi_i \restriction D) \in \hom(\mathcal{D}, \mathcal{C}_i)$, and so there is some $a_i \in (D \setminus D^{\mathcal{D}})$ such that $\pi_i(a_i) \in D^{\mathcal{C}_i}$. By induction on the cardinality of any $J \subseteq I$, let us prove that there is some $b \in (D \setminus D^{\mathcal{D}})$ such that $\pi_j(b) \in D^{\mathcal{C}_j}$, for all $j \in J$, as follows. In case $J = \varnothing$, take any $b \in (D \setminus D^{\mathcal{D}}) \neq \varnothing$, for $\mathcal{D}$ is consistent. Otherwise, take any $j \in J$, in which case $K \triangleq (J \setminus \{j\}) \subseteq I$, while $|K| < |J|$, so, by the induction hypothesis, there is some $c \in (D \setminus D^{\mathcal{D}})$ such that $\pi_k(c) \in D^{\mathcal{C}_k}$, for all $k \in K$. Then, by the $\veebar$-disjunctivity of $\mathcal{D}$, $b \triangleq (c \veebar^{\mathfrak{D}} a_j) \in (D \setminus D^{\mathcal{D}})$, while $\pi_i(b) \in D^{\mathcal{C}_i}$, for all $i \in J = (K \cup \{j\})$, because $(\pi_i \restriction D) \in \hom(\mathfrak{D}, \mathfrak{C}_i)$, while $\mathcal{C}_i$ is weakly $\veebar$-disjunctive. In particular, when $J = I$, there is some $b \in (D \setminus D^{\mathcal{D}})$ such that $\pi_i(b) \in D^{\mathcal{C}_i}$, for all $i \in I$. This contradicts to the fact that $D^{\mathcal{D}} = (D \cap \bigcap_{i \in I} \pi_i^{-1}[D^{\mathcal{C}_i}])$, as required. □

By Lemmas 2.11, 2.12, 3.19 and Remark 2.9(ii), we immediately have:

Corollary 3.20. *Let* M *be a finite class of finite weakly* $\veebar$*-disjunctive* Σ*-matrices and* $\mathcal{A}$ *a finitely-generated (in particular, finite) consistent* $\veebar$*-disjunctive model of the logic of* M*. Then,* $\mathcal{A} \in \mathbf{H}^{-1}(\mathbf{H}(\mathbf{S}_*(\mathsf{M})))$*.*

Corollary 3.21. *Let* C *be a* Σ*-logic. (Suppose it is defined by a finite class* M *of finite [weakly* $\veebar$*-disjunctive]* Σ*-matrices.) Then, (i)*$\Leftrightarrow$*(ii)*$\Leftrightarrow$*(iii)(*$\Leftrightarrow$*(iv)), where:*

(i) *C is purely-inferential;*
(ii) *C has a truth-empty model;*
(iii) *C has a one-valued truth-empty model;*
(iv) $\mathbf{P}^{\mathrm{SD}}_{\omega[\cap 0]}(\mathbf{S}_*(\mathsf{M}))[\cup \mathbf{S}_*(\mathsf{M})]$ *has a truth-empty member.*

Proof. First, (ii)$\Rightarrow$(i) is immediate. The converse is by the fact that, by the structurality of C, $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, C(\varnothing)\rangle$ is a model of C.

Next, (ii) is a particular case of (iii). Conversely, let $\mathcal{A} \in \mathrm{Mod}(C)$ be truth-empty. Then, $\chi^{\mathcal{A}}$ is singular, in which case $\theta^{\mathcal{A}} = A^2 \in \mathrm{Con}(\mathfrak{A})$, and so, by (2.23), $(\mathcal{A}/\theta^{\mathcal{A}}) \in \mathrm{Mod}(C)$ is both one-valued and truth-empty.

(Finally, (iv)$\Rightarrow$(ii) is by (2.23). Conversely, (iii)$\Rightarrow$(iv) is by Lemma 2.12 [resp., Corollary 3.20 and the $\veebar$-disjunctivity of truth-empty Σ-matrices].) ☐

Theorem 3.22. *Let* M *be a (finite) class of (finite [hereditarily simple])* Σ*-matrices,* C *the logic of* M*,* $\mathsf{S} \subseteq \mathrm{Mod}(C)$ *(the class of all* $\veebar$*-disjunctive members of* $\mathbf{S}_*(\mathsf{M})$*; cf.* (2.23)*) and* K *equal to either* $\Re[\mathsf{S}]$ *or* S *(resp., to* $((\Re(\mathsf{S})[\cap\varnothing])[\cup\mathsf{S}])$*). Then, ((i)*$\Rightarrow$*)(ii)*$\Rightarrow$*(iii)(*$\Rightarrow$*(i)), where:*

(i) *C is $\veebar$-disjunctive;*
(ii) *for each $\mathcal{A} \in \mathsf{M}$ and every $a \in (A \setminus D^{\mathcal{A}})$, there are some $\mathcal{B} \in \mathsf{K}$ and some $h \in \hom^{\{\mathrm{S}\}}(\mathcal{A},\mathcal{B})$ such that $h(a) \notin D^{\mathcal{B}}$;*
(iii) *C is defined by S.*

(In particular, any $\veebar$*-disjunctive* Σ*-logic defined by a finite class of finite* Σ*-matrices is defined by a finite class of finite* $\veebar$*-disjunctive* Σ*-matrices.)*

Proof. (First, (iii)$\Rightarrow$(i) is immediate.

Next, assume (i) holds. Consider any $\mathfrak{A} \in \mathsf{M}$ and any $a \in (A \setminus D^{\mathcal{A}})$. Then, by Corollaries 3.16(i)$\Rightarrow$(ii) and 3.17, there is some $\veebar^{\mathfrak{A}}$-disjunctive $F \in \mathrm{Fi}_C(\mathfrak{A})$ such that $D^{\mathcal{A}} \subseteq F \not\ni a$, in which case $\mathcal{D} \triangleq \langle \mathfrak{A}, F\rangle$ is a finite $\veebar$-disjunctive model of C, and so, since every member of $\mathsf{M} \subseteq \mathrm{Mod}(C)$ is weakly $\veebar$-disjunctive, for C is so, by Corollary 3.20 and Remark 2.9(ii), there are some $\mathcal{E} \in \mathsf{S}$, some Σ-matrix $\mathcal{F}$ and some $(f|g) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}|\mathcal{E},\mathcal{F})$. Hence, by Remark 2.8(i), $(\ker g) \subseteq \theta \triangleq \eth(\mathcal{E})$, in which case, by the Homomorphism Theorem, $e \triangleq (\nu_\theta \circ g^{-1}) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F},\mathcal{B})$, where $\mathcal{B} \triangleq \Re(\mathcal{E}) \in \Re[\mathsf{S}]$, and so $h \triangleq (e \circ f) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{B})$. [Likewise, by Remark 2.8(ii), g is injective, for $\mathcal{E}$ is simple, in which case $e \triangleq g^{-1} \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F},\mathcal{B})$, where $\mathcal{B} \triangleq \mathcal{E} \in \mathsf{S}$, and so $h \triangleq (e \circ f) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D},\mathcal{B})$.] In this way, $\mathcal{B} \in \mathsf{K}$, while $h \in \hom^{\mathrm{S}}(\mathcal{A},\mathcal{B})$, whereas $h(a) \notin D^{\mathcal{B}}$, for $D^{\mathcal{A}} \subseteq F = D^{\mathcal{D}} \not\ni a$. Thus, (ii) holds.)

Assume (ii) holds. Then, by (2.23), $\mathsf{K} \subseteq (\Re[\mathsf{S}] \cup \mathsf{S}) \subseteq \mathrm{Mod}(C)$. Conversely, consider any Σ-rule $\Gamma \vdash \varphi$ not satisfied in C, in which case there are some $\mathcal{A} \in \mathsf{M}$ and some $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $g[\Gamma] \subseteq D^{\mathcal{A}} \not\ni a \triangleq g(\varphi)$, and so, by (ii), there are some $\mathcal{B} \in \mathsf{K}$ and some $h \in \hom(\mathcal{A},\mathcal{B})$ such that $h(a) \notin D^{\mathcal{B}}$. Then, $f \triangleq (h \circ g) \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{B})$, while $f[\Gamma] = h[g[\Gamma]] \subseteq h[D^{\mathcal{A}}] \subseteq D^{\mathcal{B}} \not\ni h(a) = f(\varphi)$. Thus, C is defined by K, and so, by (2.23), (iii) holds. ☐

Theorem 3.22(i)$\Leftrightarrow$(ii) yields an effective algebraic criterion of the disjunctivity of finitely-valued logics.

Theorem 3.23. *Let $\mathcal{A}$ be a finite weakly $\veebar$-disjuncive Σ-matrix with unary unitary equality determinant Υ, C the logic of $\mathcal{A}$ and $\mathcal{B}$ a consistent $\veebar$-disjunctive model of C. Then,* $\hom_{\mathrm{S}}(\mathcal{B},\mathcal{A}) \neq \varnothing$.

Proof. Take any $b \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$. Consider any $F \in \wp_\omega(\{b\}, B)$. Then, by (2.23) and Remark 2.9(ii), the submatrix $\mathcal{B}_F$ of $\mathcal{B}$ generated by F is a finitely-generated consistent $\veebar$-disjunctive model of C. Therefore, by Corollary 3.20, Remark 2.8(ii) and Theorem 3.11(ii)$\Rightarrow$(i), there is some $h_F \in \hom_{\mathrm{S}}(\mathcal{B}_F, \mathcal{A})$. Now, consider any $G \in \wp_\omega(F, B) \subseteq \wp_\omega(\{b\}, B)$, in which case $B_F \subseteq B_G \subseteq B$, and any $a \in B_F$. Then, for each $\upsilon \in \Upsilon$, $(D^{\mathcal{A}} \ni \upsilon^{\mathfrak{A}}(h_F(a)) = h_F(\upsilon^{\mathfrak{B}_F}(a))) \Leftrightarrow (\upsilon^{\mathfrak{B}}(a) = \upsilon^{\mathfrak{B}_F}(a) \in D^{\mathcal{B}_F}) \Leftrightarrow (\upsilon^{\mathfrak{B}_G}(a) = \upsilon^{\mathfrak{B}}(a) \in D^{\mathcal{B}}) \Leftrightarrow (\upsilon^{\mathfrak{B}_G}(a) \in D^{\mathcal{B}_G}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(h_G(a)) = h_G(\upsilon^{\mathfrak{B}_G}(a)) \in D^{\mathcal{A}})$, in which case $h_F(a) = h_G(a)$, and so $h_F \subseteq h_G$. Therefore, $\mathcal{H} \triangleq \{h_F \mid F \in \wp_\omega(\{b\}, B)\}$ is an upward-directed (for $\wp_\omega(\{b\}, B)$ is so) subset of the inductive set of all subalgebras of $\mathfrak{B} \times \mathfrak{A}$ (uniquely determined by, and so identified with their carriers). Hence, $h \triangleq \bigcup \mathcal{H}$ forms a subalgebra of $\mathfrak{B} \times \mathfrak{A}$. And what is more, $B = \bigcup \wp_3(\{b\}, B) \subseteq \bigcup \wp_\omega(\{b\}, B) \subseteq \bigcup\{B_F \mid F \in \wp_\omega(\{b\}, B)\} \subseteq B$, in which case $(\operatorname{dom} h) = \bigcup\{\operatorname{dom} f \mid f \in \mathcal{H}\} = \bigcup\{B_F \mid F \in \wp_\omega(\{b\}, B)\} = B$, while, for all $F, G \in \wp_\omega(\{b\}, B)$, $H \triangleq (F \cup G) \in \wp_\omega(\{b\}, B)$, in which case, for every $a \in (B_F \cap B_G)$, $h_F(a) = h_H(a) = h_G(a)$, and so h is a function, whereas $(\operatorname{img} h) = \bigcup\{\operatorname{img} f \mid f \in \mathcal{H}\} \subseteq A$, and so $h : B \to A$. In this way, $h \in \hom(\mathfrak{B}, \mathfrak{A})$. Finally, consider any $a \in B$, in which case $a \in F \triangleq \{a, b\} \in \wp_\omega(\{b\}, B)$, and so $(a \in D^{\mathcal{B}}) \Leftrightarrow (a \in D^{\mathcal{B}_F}) \Leftrightarrow (D^{\mathcal{A}} \ni h_F(a) = h(a))$. Thus, $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$. ☐

3.4. **Implicativity.** Fix any set A, any closure operator C over A and any $\iota : A^2 \to A$, in which case we put $\delta_\iota(a,b) : A^2 \to A, \langle a, b\rangle \mapsto \iota(\iota(a,b),b)$.

Next, C is said to *have Abstract Deduction Theorem (ADT) with respect to* ι, provided, for all $a \in X \subseteq A$ and all $b \in C(X)$, it holds that $\iota(a,b) \in C(X \setminus \{a\})$. Then, C is said to be *weakly ι-implicative*, provided it has ADT with respect to ι and

$$b \in C(\{a, \iota(a,b)\}), \tag{3.9}$$

for all $a, b \in A$. Likewise, C is said to be *(strongly) ι-implicative*, whenever it is weakly so and

$$\delta_\iota(\iota(a,b),a) \in C(\varnothing), \tag{3.10}$$

for all $a, b \in A$.

Lemma 3.24. *Suppose C is ι-implicative. Then, it is δ_ι-disjunctive.*

Proof. With using Lemma 3.14. Consider, any $(X \cup \{a,b\}) \subseteq A$. Then, (3.4) is by ADT w.r.t. ι. Next, (3.5) is by (3.9) and (3.10). Further, by (3.9) and ADT w.r.t. ι, we have $\iota(\iota(a,b),a) \in C(\{\iota(b,a), \delta_\iota(a,b)\})$, in which case, by (3.9) and (3.10), we get $a \in C(\{\iota(b,a), \delta_\iota(a,b)\})$, and so, by ADT w.r.t. ι, we eventually get (3.6). Finally, consider any $c \in C(X \cup \{b\})$. Then, by (3.9) and ADT w.r.t. ι, we have $\iota(b,a) \in C(X \cup \{\iota(c,a)\})$, in which case, by (3.9), we get $a \in C(X \cup \{\delta_\iota(b,a), \iota(c,a)\})$, and so, by ADT w.r.t. ι, we eventually get $\delta_\iota(c,a) \in C(X \cup \{\delta_\iota(b,a)\})$. Thus, (3.7) holds, as required. ☐

3.4.1. *Implicative matrices versus implicative logics.*

Lemma 3.25. *Let C be an $\sqsupset$-implicative Σ-logic and $\mathcal{A}$ a $\uplus_\sqsupset$-disjunctive model of C. Then, $\mathcal{A}$ is $\sqsupset$-implicative.*

Proof. By the fact that (2.11), (2.13) and (2.15) $= ((x_0 \sqsupset x_1) \uplus_\sqsupset x_0)$, being satisfied in C, are true in $\mathcal{A}$. ☐

Combining Lemmas 3.24, 3.25 with Theorem 3.18, we first have:

Corollary 3.26. *A [finitary] Σ-logic is $\sqsupset$-implicative if[f] it is defined by a class of $\sqsupset$-implicative Σ-matrices.*

Likewise, combining (2.23), Lemmas 3.24, 3.25 with Theorem 3.22, we also have:

Corollary 3.27. *Let M be a finite class of finite [hereditarily simple] Σ-matrices, C the logic of M, S the class of all $\sqsupset$-implicative members of $\mathbf{S}_*(\mathsf{M})$ and $\mathsf{K} \triangleq ((\Re(\mathsf{S})[\cap\varnothing])[\cup\mathsf{S}])$. Then, the following are equivalent:*

(i) *C is $\sqsupset$-implicative;*

(ii) *for each $\mathcal{A} \in \mathsf{M}$ and every $a \in (A \setminus D^{\mathcal{A}})$, there are some $\mathcal{B} \in \mathsf{K}$ and some $h \in \hom^{(\mathrm{S})}(\mathcal{A}, \mathcal{B})$ such that $h(a) \notin D^{\mathcal{B}}$;*

(iii) *C is defined by S.*

In particular, any $\sqsupset$-implicative Σ-logic defined by a finite class of finite Σ-matrices is defined by a finite class of finite $\sqsupset$-implicative Σ-matrices.

Corollary 3.27(i)$\Leftrightarrow$(ii) yields an effective algebraic criterion of the implicativity of finitely-valued logics.

3.4.2. *Implicative calculi versus implicative logics.*

Lemma 3.28. *Let C' be a finitary Σ-logic and C'' a 1-extension of C'. Suppose C' has DT with respect to $\sqsupset$, while* (2.11) *is satisfied in C''. Then, C'' is an extension of C'. In particular, any exiomatically-equivalent finitary weakly $\sqsupset$-implicative Σ-logics are equal.*

Proof. By induction on any $n \in \omega$, we prove that C'' is an n-extension of C'. For consider any $X \in \wp_n(\mathrm{Fm}^{\omega}_{\Sigma})$, in which case $n \neq 0$, and any $\psi \in C'(X)$. Then, in case $X = \varnothing$, we have $X \in \wp_1(\mathrm{Fm}^{\omega}_{\Sigma})$, and so $\psi \in C'(X) \subseteq C''(X)$, for C'' is a 1-extension of C'. Otherwise, take any $\phi \in X$, in which case $Y \triangleq (X \setminus \{\phi\}) \in \wp_{n-1}(\mathrm{Fm}^{\omega}_{\Sigma})$, and so, by DT with respect to $\sqsupset$, that C' has, and the induction hypothesis, we have $(\phi \sqsupset \psi) \in C'(Y) \subseteq C''(Y)$. Therefore, by (2.11)$[x_0/\phi, x_1/\psi]$ satisfied in C'', in view of its structurality, we eventually get $\psi \in C''(Y \cup \{\phi\}) = C''(X)$. Hence, since $\omega = (\bigcup \omega)$, we eventually conclude that C'' is an ω-extension of C', and so an extension of C', for this is finitary. □

By $\mathfrak{I}^{[\mathrm{PL}]}_{\sqsupset}$ we denote the Σ-calculus constituted by (2.11), (2.13) and (2.14) [as well as (2.15)].

Lemma 3.29 (cf. Theorem 2.5 of [15])**.** *Let $\mathcal{A}$ be an axiomatic Σ-calculus, C' the Σ-logic axiomatized by $\mathfrak{I}_{\sqsupset} \cup \mathcal{A}$ and $\mathfrak{A}$ a Σ-algebra. Then, $\mathrm{Fg}^{\mathfrak{A}}_{C'}$ has ADT with respect to $\sqsupset^{\mathfrak{A}}$.*

Proof. Consider any $a \in X \subseteq A$ and any $b \in \mathrm{Fg}^{\mathfrak{A}}_{C'}(X)$, in which case there is some $(\mathfrak{I}_{\sqsupset} \cup \mathcal{A})$-derivation $\bar{c}$ of b from X over $\mathfrak{A}$. Then, by induction on any $i \in (\mathrm{dom}\,\bar{c})$, with using the derivability of (2.12) in $\mathfrak{I}_{\sqsupset}$ and Herbrand's method (cf., e.g., the proof of Proposition 1.8 of [12]), it is routine checking that $(a \sqsupset^{\mathfrak{A}} c_i) \in \mathrm{Fg}^{\mathfrak{A}}_{C'}(X \setminus \{a\})$. In this way, the fact that $b \in (\mathrm{img}\,\bar{c})$ completes the argument. □

Corollary 3.30. *Finitary weakly $\sqsupset$-implicative Σ-logics are exactly axiomatic extensions of the Σ-logic axiomatized by $\mathfrak{I}_{\sqsupset}$.*

Proof. Let C' be a finitary $\sqsupset$-implicative Σ-logic and C'' the Σ-logic axiomatized by $\mathfrak{I}_{\sqsupset} \cup C'(\varnothing)$. Then, C' is an extension of C''. Conversely, C'' is a 1-extension of C', and so, by Lemma 3.28, is an extension of C'. In this way, Lemma 3.29 with $\mathfrak{A} = \mathfrak{Fm}^{\omega}_{\Sigma}$ completes the argument. □

After all, combining Lemma 3.29 and Corollary 3.30, we immediately get:

Corollary 3.31. *Let C' be a finitary [weakly] $\sqsupset$-implicative Σ-logic and $\mathfrak{A}$ a Σ-algebra. Then, $\mathrm{Fg}^{\mathfrak{A}}_{C'}$ is [weakly] $\sqsupset^{\mathfrak{A}}$-implicative.*

3.5. **Classical matrices and logics.** A two-valued Σ-matrix $\mathcal{A}$ is said to be $\sim$-*classical*, whenever it is $\sim$-negative, in which case it is both consistent and truth-non-empty, and so is both false- and truth-singular, the unique element of $(A \setminus D^{\mathcal{A}})/D^{\mathcal{A}}$ being denoted by $(0/1)_{\mathcal{A}}$, respectively (the index $_{\mathcal{A}}$ is often omitted, unless any confusion is possible), in which case $A = \{0, 1\}$, while $\sim^{\mathfrak{A}} i = (1 - i)$, for each $i \in 2$, whereas $\theta^{\mathcal{A}}$ is diagonal, for $\chi^{\mathcal{A}}$ is so, and so $\mathcal{A}$ is simple (in particular, hereditarily so, for it has no proper submatrix) but is not $\sim$-paraconsistent, in view of Remark 2.9(i)**d)**.

A Σ-logic is said to be $\sim$-*[sub]classical*, whenever it is [a sublogic of] the logic of a $\sim$-classical Σ-matrix, in which case it is inferentially consistent. Then, $\sim$ is called a *subclassical negation for* a Σ-logic C, whenever the $\sim$-fragment of C is $\sim$-subclassical, in which case:

$$\sim^m x_0 \notin C(\sim^n x_0), \tag{3.11}$$

for all $m, n \in \omega$ such that the integer $m - n$ is odd.

Lemma 3.32. *Let $\mathcal{A}$ be a $\sim$-classical Σ-matrix, C the logic of $\mathcal{A}$ and $\mathcal{B}$ a truth-non-empty consistent model of C. Then, $\mathcal{A}$ is a strict surjective homomorphic image of a submatrix of $\mathcal{B}$, in which case $\mathcal{A}$ is isomorphic to any $\sim$-classical model of C, and so C has no proper $\sim$-classical extension.*

Proof. Take any $a \in D^{\mathcal{B}} \neq \varnothing$ and any $b \in (B \setminus D^{\mathcal{B}}) \neq \varnothing$. Then, by (2.23), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b\}$ is a finitely-generated consistent truth-non-empty model of C. Therefore, by Corollary 3.20, there are some set I, some submatrix $\mathcal{E}$ of $\mathcal{A}^I$, some Σ-matrix $\mathcal{F}$, some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{F})$ and some $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{F})$, in which case $\mathcal{E}$ is both truth-non-empty and consistent (in particular, $I \neq \varnothing$), for $\mathcal{D}$ is so, and so there is some $d \in D^{\mathcal{E}} \neq \varnothing$, in which case $E \ni d \triangleq (I \times \{1\})$, and so $E \ni \sim^{\mathfrak{E}} d = (I \times \{0\})$. Hence, as $I \neq \varnothing$, $e \triangleq \{\langle x, (I \times \{x\})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$, in which case $f \triangleq (h \circ e) \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{F})$ is injective, in view of Remark 2.8(ii). Then, $G \triangleq (\operatorname{img} f)$ forms a subalgebra of $\mathfrak{F}$, in which case $H \triangleq g^{-1}[G]$ forms a subalgebra of $\mathfrak{D}$, and so $f^{-1} \circ (g \restriction G)$ is a strict surjective homomorphism from $(\mathcal{D} \restriction H) \in \mathbf{S}(\mathcal{B})$ onto $\mathcal{A}$. In this way, (2.23), Remark 2.8(ii) and the fact that any $\sim$-classical Σ-matrix is simple and has no proper submatrix complete the argument. $\square$

A $\sim$-classical Σ-matrix $\mathcal{A}$ is said to be *canonical*, whenever $A = 2$ and $a_{\mathcal{A}} = a$, for all $a \in A$, any isomorphism between canonical ones being clearly diagonal, so any isomorphic canonical ones being equal. In general, the bijection $e_{\mathcal{A}} \triangleq \{\langle i, i_{\mathcal{A}}\rangle \mid i \in 2\} : 2 \to A$ is an isomorphism from the canonical $\sim$-classical Σ-matrix $\langle e_{\mathcal{A}}^{-1}[\mathfrak{A}], \{1\}\rangle$ onto $\mathcal{A}$. In this way, in view of (2.23) and Lemma 3.32, any $\sim$-classical Σ-logic is defined by a unique canonical $\sim$-classical Σ-matrix, said to be *characteristic for/of* the logic.

Corollary 3.33. *Any $\sim$-classical Σ-logic has no proper inferentially consistent extension, and so is structurally complete iff it has a theorem.*

Proof. Let $\mathcal{A}$ be a $\sim$-classical Σ-matrix, C the logic of $\mathcal{A}$ and C' an inferentially consistent extension of C. Then, $x_1 \notin T \triangleq C'(x_0) \ni x_0$. On the other hand, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a consistent truth-non-empty model of C' (in particular, of C). In this way, (2.23), Remark 2.5 and Lemma 3.32 complete the argument. $\square$

4. Structural completions versus free models

Let M be a class of Σ-matrices, C the logic of M, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and $\alpha \in \wp_{\omega[\setminus 1]}(\omega)$ [unless Σ has a nullary symbol]. Then, for any $\mathfrak{A} \in \mathsf{M}$ and any $h \in \hom(\mathrm{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$,

$h \in \mathrm{hom_S}(\mathcal{B}, \mathcal{A})$, where $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{\alpha}_{\Sigma}, h^{-1}[D^{\mathcal{A}}] \rangle$, in which case, by Remark 2.8, we have $\theta^{\alpha}_{\mathsf{K}} \subseteq (\ker h) = h^{-1}[\Delta_A] \subseteq h^{-1}[\theta^{\mathcal{A}}] = \theta^{\mathcal{B}}$, and so $\theta^{\alpha}_{\mathsf{K}} \subseteq \theta^{\mathcal{D}}$, where $\mathcal{D} \triangleq \langle \mathfrak{Fm}^{\alpha}_{\Sigma}, \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\varnothing) \rangle \in \mathrm{Mod}(C)$, in view of the structurality of C. Thus, $\theta^{\alpha}_{\mathsf{K}} \in \mathrm{Con}(\mathcal{D})$, in which case, by (2.23), $\mathcal{F}^{\alpha}_{\mathsf{M}} \triangleq (\mathcal{D}/\theta^{\alpha}_{\mathsf{K}}) \in \mathrm{Mod}(C)$, while $\mathfrak{F}^{\alpha}_{\mathsf{M}} = \mathfrak{F}^{\alpha}_{\mathsf{K}}$.

Theorem 4.1. *Let Σ be a signature [with(out) nullary symbols], M a [finite (non-empty)] class of [finite] Σ-matrices, C the logic of M, [$f \in \prod_{\mathcal{A} \in \mathsf{M}} \wp_{\omega(\setminus 1)}(A)$] $\alpha \triangleq (\omega[\cap \bigcup_{\mathcal{A} \in \mathsf{M}} |f(\mathcal{A})|])$ and $\mathcal{B}$ a submatrix of $\mathcal{F}^{\alpha}_{\mathsf{M}}$. Suppose every $\mathcal{A} \in \mathsf{M}$ is a surjective homomorphic image of $\mathcal{B}$, unless $\mathcal{B} = \mathcal{F}^{\alpha}_{\mathsf{M}}$, [and generated by $f(\mathcal{A})$]. Then, the structural completion of C is defined by $\mathcal{B}$.*

Proof. Then, by (2.23), the logic C' of $\mathcal{F}^{\omega[/\alpha]}_{\mathsf{M}}$ is defined by $\mathcal{D}_{\omega[/\alpha]} \triangleq \langle \mathfrak{Fm}^{\omega[/\alpha]}_{\Sigma}, \mathrm{Cn}^{\omega[/\alpha]}_{\mathsf{M}}(\varnothing) \rangle \in \mathrm{Mod}(C)$, in view of the structurality of C [/and (2.22)], in which case it is an extension of C, and so $C(\varnothing) \subseteq C'(\varnothing)$. For proving the converse inclusion, consider the following complementary cases:

- $\alpha = \omega$.
  Then, applying the diagonal Σ-substitution, we get $C'(\varnothing) \subseteq D^{\mathcal{D}_\omega} = C(\varnothing)$.
- $\alpha \neq \omega$.
  Consider any $\mathcal{A} \in \mathsf{M}$, in which case it is generated by $f(\mathcal{A})$ of cardinality $\leqslant \alpha$, and so there is some surjective $h \in \mathrm{hom}(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$. Then, $D^{\mathcal{D}_\alpha} = \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\varnothing) \subseteq h^{-1}[D^{\mathcal{A}}]$, in which case $h \in \mathrm{hom}^{\mathrm{S}}(\mathcal{D}_\alpha, \mathcal{A})$, and so, by (2.24), $C'(\varnothing) \subseteq C(\varnothing)$.

Next, $\mathcal{D}_\omega$ is a model of any extension C'' of C' such that $C''(\varnothing) = C(\varnothing)$, in view of its structurality [and so is its submatrix $\mathcal{D}_\alpha$, in view of (2.22) and (2.23)], in which case C' is the structural completion of C. Finally, by (2.23), $\mathcal{B}$ is a model of C'. Conversely, if $\mathcal{B} = \{\neq\}\mathcal{F}^{\alpha}_{\mathsf{M}}$, then {each $\mathcal{A} \in \mathsf{M}$ is a surjective homomorphic image of $\mathcal{B}$, in which case, by (2.24)} $\mathrm{Cn}_{\mathcal{B}}(\varnothing) = C'(\varnothing)$, and so C', being structurally complete, is defined by $\mathcal{B}$, as required. □

The []-optional case of this theorem provides an effective procedure of finding finite matrix semantics of any finitely-valued logic, applications of which are demonstrated in proving Theorem 6.11 below.

5. Self-extensional logics versus simple matrices

Theorem 5.1. *Let M be a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$, $\mathsf{V} \triangleq \mathbf{V}(\mathsf{K})$, $\alpha \triangleq (1 \cup (\omega \cap \bigcup\{|A| \mid \mathcal{A} \in \mathsf{M}\})) \in \wp_{\infty\setminus 1}(\omega)$ and C the logic of M. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *$\equiv^{\omega}_{C} \subseteq \theta^{\omega}_{\mathsf{K}}$;*
(iii) *$\equiv^{\omega}_{C} = \theta^{\omega}_{\mathsf{K}}$;*
(iv) *for all distinct $a, b \in F^{\alpha}_{\mathsf{V}}$, there are some $\mathcal{A} \in \mathsf{M}$ and some $h \in \mathrm{hom}(\mathfrak{F}^{\alpha}_{\mathsf{V}}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$;*
(v) *there is some class C of Σ-algebras such that $\mathsf{K} \subseteq \mathbf{V}(\mathsf{C})$ and, for each $\mathfrak{A} \in \mathsf{C}$ and all distinct $a, b \in A$, there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \mathrm{hom}(\mathfrak{A}, \mathfrak{B})$ such that $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$;*
(vi) *there is some $\mathsf{S} \subseteq \mathrm{Mod}(C)$ such that $\mathsf{K} \subseteq \mathbf{V}(\pi_0[\mathsf{S}])$ and, for each $\mathcal{A} \in \mathsf{S}$, it holds that $(A^2 \cap \bigcap\{\theta^{\mathcal{B}} \mid \mathcal{B} \in \mathsf{S}, \mathfrak{B} = \mathfrak{A}\}) \subseteq \Delta_A$.*

Proof. First, (i/ii)⇒(ii/iii) is by Corollary 3.4/Lemma 3.7, respectively.

Next, assume (iii) holds. Then, $\theta^{\beta} \triangleq \equiv^{\beta}_{C} = \theta^{\beta}_{\mathsf{K}} = \theta^{\beta}_{\mathsf{V}} \in \mathrm{Con}(\mathfrak{Fm}^{\beta}_{\Sigma})$, for all $\beta \in \wp_{\infty\setminus 1}(\omega)$. In particular (when $\beta = \omega$), (i) holds. Furthermore, consider any distinct $a, b \in F^{\alpha}_{\mathsf{V}}$. Then, there are some $\phi, \psi \in \mathrm{Fm}^{\alpha}_{\Sigma}$ such that $\nu_{\theta^\alpha}(\phi) = a \neq b = \nu_{\theta^\alpha}(\phi)$, in which case, by (2.22), $\mathrm{Cn}^{\alpha}_{\mathsf{M}}(\phi) \neq \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\psi)$, and so there are some $\mathcal{A} \in \mathsf{M}$ and some $g \in \hom(\mathfrak{Fm}_\Sigma^\alpha, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(g(\phi)) \neq \chi^{\mathcal{A}}(g(\phi))$. In that case, $\theta^\alpha \subseteq (\ker g)$, and so, by the Homomorphism Theorem, $h \triangleq (g \circ \nu_{\theta^\alpha}^{-1}) \in \hom(\mathfrak{F}_{\mathsf{V}}^\alpha, \mathfrak{A})$. Then, $h(a/b) = g(\phi/\psi)$, in which case $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$, and so (iv) holds.

Further, assume (iv) holds. Let $\mathsf{C} \triangleq \{\mathfrak{F}_{\mathsf{V}}^\alpha\}$. Consider any $\mathfrak{A} \in \mathsf{K}$ and the following complementary cases:

- $|A| \leqslant \alpha$.
  Let $h \in \hom(\mathfrak{Fm}_\Sigma^\alpha, \mathfrak{A})$ extend any surjection from V_α onto A, in which case it is surjective, while $\theta \triangleq \theta_{\mathsf{V}}^\alpha = \theta_{\mathsf{K}}^\alpha \subseteq (\ker h)$, and so, by the Homomorphism Theorem, $g \triangleq (h \circ \nu_\theta^{-1}) \in \hom(\mathfrak{F}_{\mathsf{V}}^\alpha, \mathfrak{A})$ is surjective. In this way, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}_{\mathsf{V}}^\alpha)$.
- $|A| \nleqslant \alpha$.
  Then, $\alpha = \omega$. Consider any Σ-identity $\phi \approx \psi$ true in $\mathfrak{F}_{\mathsf{V}}^\omega$ and any $h \in \hom(\mathfrak{Fm}_\Sigma^\omega, \mathfrak{A})$, in which case, we have $\theta \triangleq \theta_{\mathsf{V}}^\omega = \theta_{\mathsf{K}}^\omega \subseteq (\ker h)$, and so, since $\nu_\theta \in \hom(\mathfrak{Fm}_\Sigma^\omega, \mathfrak{F}_{\mathsf{V}}^\omega)$, we get $\langle \phi, \psi \rangle \in (\ker \nu_\theta) \subseteq (\ker h)$. In this way, $\mathfrak{A} \in \mathbf{V}(\mathfrak{F}_{\mathsf{V}}^\alpha)$.

Thus, $\mathsf{K} \subseteq \mathbf{V}(\mathsf{C})$, and so (v) holds.

Now, assume (v) holds. Let C' be the class of all non-one-element members of C and $\mathsf{S} \triangleq \{\langle \mathfrak{A}, h^{-1}[D^{\mathcal{B}}]\rangle \mid \mathfrak{A} \in \mathsf{C}', \mathcal{B} \in \mathsf{M}, h \in \hom(\mathfrak{A}, \mathfrak{B})\}$. Then, for all $\mathfrak{A} \in \mathsf{C}'$, each $\mathcal{B} \in \mathsf{M}$ and every $h \in \hom(\mathfrak{A}, \mathfrak{B})$, h is a strict homomorphism from $\mathcal{C} \triangleq \langle \mathfrak{A}, h^{-1}[D^{\mathcal{B}}]\rangle$ to $\mathcal{B}$, in which case, by (2.23), $\mathcal{C} \in \mathrm{Mod}(C)$, and so $\mathsf{S} \subseteq \mathrm{Mod}(C)$, while $\chi^{\mathcal{C}} = (\chi^{\mathcal{B}} \circ h)$, whereas $\pi_0[\mathsf{S}] = \mathsf{C}'$ generates the variety $\mathbf{V}(\mathsf{C})$. In this way, (vi) holds.

Finally, assume (vi) holds. Consider any $\phi, \psi \in \mathrm{Fm}_\Sigma^\omega$ such that $\phi \equiv_C^\omega \psi$, any $\mathcal{A} \in \mathsf{S}$ and any $h \in \hom(\mathfrak{Fm}_\Sigma^\omega, \mathfrak{A})$. Then, for each $\mathcal{B} \in \mathsf{S}$ with $\mathfrak{B} = \mathfrak{A}$, $h(\phi)\ \theta^{\mathcal{B}}\ h(\psi)$, in which case $h(\phi) = h(\psi)$, and so $\mathfrak{A} \models (\phi \approx \psi)$. Thus, $\mathsf{K} \subseteq \mathbf{V}(\pi_0[\mathsf{S}]) \models (\phi \approx \psi)$, and so (ii) holds, as required. □

When both M and all members of it are finite, α is finite, in which case $\mathfrak{F}_{\mathsf{V}}^\alpha$ is finite and can be found effectively, and so, taking (2.23) and Remark 2.8[(iv)] into account, the item (iv) of Theorem 5.1 yields an effective procedure of checking the self-extensionality of any logic defined by a finite class of finite matrices. However, its computational complexity may be too large to count it *practically* applicable. For instance, in the unitary n-valued case, where $n \in \omega$, the upper limit n^{n^n} of $|F_{\mathsf{V}}^\alpha|$ as well as the predetermined computational complexity $n^{n^{n^n}}$ of the procedure involved become too large even in the three-/four-valued case. And, though, in the two-valued case, this limit — 16 — as well as the respective complexity — $2^{16} = 65536$ — are reasonably acceptable, this is no longer matter in view of the following universal observation:

Example 5.2. Let $\mathcal{A}$ be a Σ-matrix. Suppose it is both false- and truth-singular (in particular, two-valued as well as both consistent and truth-non-empty [in particular, classical]), in which case $\theta^{\mathcal{A}} = \Delta_A$, for $\chi^{\mathcal{A}}$ is injective, and so $\mathcal{A}$ is simple. Then, by Theorems 3.9 and 5.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}\}$, the logic of $\mathcal{A}$ is self-extensional, its intrinsic variety being generated by $\mathfrak{A}$. Thus, by the self-extensionality of inferentially inconsistent logics, *any two-valued (in particular, classical) logic is self-extensional.* □

Nevertheless, the procedure involved is simplified much under certain conditions upon the basis of the item (v) of Theorem 5.1.

5.1. Self-extensional conjunctive disjunctive logics.

Lemma 5.3. *Let C be a [finitary $\overline{\wedge}$-conjunctive] Σ-logic and $\mathcal{A}$ a [truth-non-empty $\overline{\wedge}$-conjunctive] Σ-matrix. Then, $\mathcal{A} \in \mathrm{Mod}_{2\backslash 1}(C)$ if[f] $\mathcal{A} \in \mathrm{Mod}(C)$.*

Proof. The "if" part is trivial. [Conversely, assume $\mathcal{A} \in \mathrm{Mod}_{2\setminus 1}(C)$. Then, by Remark 2.6, $\mathcal{A} \in \mathrm{Mod}_2(C)$. By induction on any $n \in \omega$, let us prove that $\mathcal{A} \in \mathrm{Mod}_n(C)$. For consider any $X \in \wp_n(\mathrm{Fm}^\omega_\Sigma)$, in which case $n \neq 0$. In case $|X| \in 2$, $X \in \wp_2(\mathrm{Fm}^\omega_\Sigma)$, and so $C(X) \subseteq \mathrm{Cn}^\omega_{\mathcal{A}}(X)$, for $\mathcal{A} \in \mathrm{Mod}_2(C)$. Otherwise, $|X| \geqslant 2$, in which case there are some distinct $\phi, \psi \in X$, and so $Y \triangleq ((X \setminus \{\phi,\psi\}) \cup \{\phi \mathbin{\bar\wedge} \psi\}) \in \wp_{n-1}(\mathrm{Fm}^\omega_\Sigma)$. Then, by the induction hypothesis and the $\bar\wedge$-conjunctivity of both C and $\mathcal{A}$, we get $C(X) = C(Y) \subseteq \mathrm{Cn}^\omega_{\mathcal{A}}(Y) = \mathrm{Cn}^\omega_{\mathcal{A}}(X)$. Thus, $\mathcal{A} \in \mathrm{Mod}_\omega(C)$, for $\omega = (\bigcup \omega)$, and so $\mathcal{A} \in \mathrm{Mod}(C)$, for C is finitary.] □

Remark 5.4. Let C be a $\bar\wedge$-conjunctive or/and $\veebar$-disjunctive Σ-logic and $\phi \approx \psi$ a semi-lattice/"distributive lattice" identity for $\bar\wedge$ or/and $\veebar$, respectively. Then, $\phi \equiv^\omega_C \psi$. □

Theorem 5.5. *Let* M *be a class of simple* Σ*-matrices,* $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$*,* $\mathsf{V} \triangleq \mathbf{V}(\mathsf{K})$ *and* C *the logic of* M*. Suppose* C *is finitary (in particular, both* M *and all members of it are finite) and* $\bar\wedge$*-conjunctive (that is, all members of* M *are so) [as well as* $\veebar$*-disjunctive (in particular, all members of* M *are so)]. Then, the following are equivalent:*

(i) C *is self-extensional;*
(ii) *for all* $\phi, \psi \in \mathrm{Fm}^\omega_\Sigma$*, it holds that* $(\psi \in C(\phi)) \Leftrightarrow (\mathsf{K} \models (\phi \approx (\phi \mathbin{\bar\wedge} \psi)))$*, while semi-lattice [resp., distributive lattice] identities for* $\bar\wedge$ *[and* $\veebar$*] are true in* K*;*
(iii) *every truth-non-empty* $\bar\wedge$*-conjunctive* Σ*-matrix with underlying algebra in* V *is a model of* C*, while semi-lattice [resp., distributive lattice] identities for* $\bar\wedge$ *[and* $\veebar$*] are true in* V*;*
(iv) *any truth-non-empty* $\bar\wedge$*-conjunctive [consistent* $\veebar$*- disjunctive]* Σ*-matrix with underlying algebra in* K *is a model of* C*, while semi-lattice [resp., distributive lattice] identities for* $\bar\wedge$ *[and* $\veebar$*] are true in* K*.*

Proof. First, (i)⇒(ii) is by Theorem 5.1(i)⇒(iii), Remark 5.4 and the $\bar\wedge$-conjuctivity of C. Next, (ii)⇒(iii) is by Lemma 5.3. Further, (iv) is a particular case of (iii). Finally, (iv)⇒(i) is by Theorem 5.1(vi)⇒(i) with S, being the class of all truth-non-empty $\bar\wedge$-conjunctive [consistent $\veebar$- disjunctive] Σ-matrices with underlying algebra in K, and the semilattice identities for $\bar\wedge$ [as well as the Prime Ideal Theorem for distributive lattices]. (More precisely, consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar a \in (A^2 \setminus \Delta_A)$, in which case, by the semilattice identities (more specifically, the commutativity one) for $\bar\wedge$, $a_i \neq (a_i \mathbin{\bar\wedge^{\mathfrak{A}}} a_{1-i})$, for some $i \in 2$, and so $\mathcal{B} \triangleq \langle \mathfrak{A}, \{b \in A \mid a_i = (a_i \mathbin{\bar\wedge^{\mathfrak{A}}} b)\}\rangle \in \mathsf{S}$ [resp., by the Prime Ideal Theorem, there is some $\mathcal{B} \in \mathsf{S}$] such that $\mathfrak{B} = \mathfrak{A}$ and $a_i \in D^{\mathcal{B}} \not\ni a_{1-i}$.) □

Theorem 5.6. *Let* M *be a finite class of finite hereditarily simple* $\bar\wedge$*-conjunctive* $\veebar$*-disjunctive* Σ*-matrices,* $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ *and* C *the logic of* M*. Then,* C *is self-extensional iff, for each* $\mathfrak{A} \in \mathsf{K}$ *and all distinct* $a, b \in A$*, there are some* $\mathcal{B} \in \mathsf{M}$ *and some* $h \in \hom(\mathfrak{A}, \mathfrak{B})$ *such that* $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$*.*

Proof. The "if" part is by Theorem 5.1(v)⇒(i) with $\mathsf{C} = \mathsf{K}$. Conversely, assume C is self-extensional. Consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar a \in (A^2 \setminus \Delta_A)$. Then, by Theorem 5.5(i)⇒(iv), $\mathfrak{A}$ is a distributive $(\bar\wedge, \veebar)$-lattice, in which case, by the commutativity identity for $\bar\wedge$, $a_i \neq (a_i \mathbin{\bar\wedge^{\mathfrak{A}}} a_{1-i})$, for some $i \in 2$, and so, by the Prime Ideal Theorem, there is some $\bar\wedge$-conjunctive $\veebar$-disjunctive Σ-matrix $\mathcal{D}$ with $\mathfrak{D} = \mathfrak{A}$ such that $a_i \in D^{\mathcal{D}} \not\ni a_{1-i}$, in which case $\mathcal{D}$ is both consistent and truth-non-empty, and so is a model of C. Hence, by Corollary 3.20 and Remark 2.8(ii), there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B}) \subseteq \hom(\mathfrak{A}, \mathfrak{B})$, in which case $h(a_i) \in D^{\mathcal{B}} \not\ni h(a_{1-i})$, and so $\chi^{\mathcal{B}}(h(a_i)) = 1 \neq 0 = \chi^{\mathcal{B}}(h(a_{1-i}))$, as required. □

5.2. **Self-extensional implicative logics.**

Lemma 5.7. *Let C be a Σ-logic, $\mathcal{A} \in \mathrm{Mod}^*(C)$ and $\phi, \psi \in C(\varnothing)$. Suppose C is self-extensional. Then, $\mathfrak{A} \models (\phi \approx \psi)$.*

Proof. In that case, $\phi \equiv^{\omega}_{C} \psi$, and so Corollary 3.5 completes the argument. □

Lemma 5.8. *Let C be a Σ-logic, $\mathcal{A} \in \mathrm{Mod}^*(C)$, $a \in A$ and $\mathcal{B} \triangleq \langle \mathfrak{A}, \{a \sqsupset^{\mathfrak{A}} a\}\rangle$. Suppose C is finitary, self-extensional and weakly $\sqsupset$-implicattive. Then, $(a \sqsupset^{\mathfrak{A}} a) \sqsupset^{\mathfrak{A}} b) = b$, for all $b \in A$, in which case $\mathcal{B} \in \mathrm{Mod}(C)$, and so $D^{\mathcal{B}} = \mathrm{Fg}^{\mathfrak{A}}_{C}(\varnothing)$.*

Proof. Let $\varphi \in C(\varnothing)$ and $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$. Then, $V \triangleq \mathrm{Var}(\phi) \in \wp_{\omega}(V_{\omega})$, in which case $(V_{\omega} \setminus V) \neq \varnothing$, and so there is some $v \in (V_{\omega} \setminus V)$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ extend $(h \upharpoonright (V_{\omega} \setminus \{v\})) \cup [v/a]$. Then, as, by (2.12), $(v \sqsupset v) \in C(\varnothing)$, by Lemma 5.7, we have $h(\varphi) = g(\varphi) = g(v \sqsupset v) = (a \sqsupset^{\mathfrak{A}} a) \in D^{\mathcal{B}}$, and so $\mathcal{B} \in \mathrm{Mod}_1(C)$. Moreover, as, by (2.12), $(x_0 \sqsupset x_0) \in C(\varnothing)$, by (2.13) and (2.11), we have $((x_0 \sqsupset x_0) \sqsupset x_1) \equiv^{\omega}_{C} x_1$, in which case, by Corollary 3.5, we get $(a \sqsupset^{\mathfrak{A}} a) \sqsupset^{\mathfrak{A}} b) = b$, for all $b \in A$, and so (2.11) is true in $\mathcal{B}$. In this way, (2.12) and Lemma 3.28 complete the argument. □

Theorem 5.9. *Let M be a class of simple Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Suppose C is finitary (in particular, both M and all members of it are finite) and $\sqsupset$-implicative (in particular, all members of M are so). Then, C is self-extensional iff, for all $\phi, \psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, it holds that $(\psi \in C(\phi)) \Leftrightarrow (\mathsf{K} \models (\psi \approx (\phi \uplus_{\sqsupset} \psi)))$, while both (2.3) and (2.4) as well as semi-lattice identities for $\uplus_{\sqsupset}$ are true in K.*

Proof. The "if" part is by Theorem 5.1(ii)⇒(i) and semi-lattice identities (more specifically, the commutativity one) for $\uplus_{\sqsupset}$. Conversely, by Lemma 3.24, C is $\uplus_{\sqsupset}$-disjunctive. In this way, Theorem 5.1(i)⇒(iii), Remark 5.4, (2.12), Lemma 5.8 and the $\uplus_{\sqsupset}$-disjunctivity of C complete the argument. □

Now, we are in a position to prove the following "implicative" analogue of Theorem 5.6:

Theorem 5.10. *Let M be a finite class of finite hereditarily simple $\sqsupset$-implicative Σ-matrices, $\mathsf{K} \triangleq \pi_0[\mathsf{M}]$ and C the logic of M. Then, C is self-extensional iff, for each $\mathfrak{A} \in \mathsf{K}$ and all distinct $a, b \in A$, there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom(\mathfrak{A}, \mathfrak{B})$ such that $\chi^{\mathcal{B}}(h(a)) \neq \chi^{\mathcal{B}}(h(b))$.*

Proof. The "if" part is by Theorem 5.1(v)⇒(i) with $\mathsf{C} = \mathsf{K}$. Conversely, assume C is self-extensional. Consider any $\mathfrak{A} \in \mathsf{K}$ and any distinct $a, b \in A$. Then, by Theorem 5.9, $\mathfrak{A}$ is a $\uplus_{\sqsupset}$-semi-lattice satisfying (2.4), in which case, by the commutativity identity for $\uplus_{\sqsupset}$, without loss of generality, $b \neq (a \uplus^{\mathfrak{A}}_{\sqsupset} b)$, and so $b \notin \mathrm{Fg}^{\mathfrak{A}}_{C}(a)$, for, otherwise, by Corollary 3.31 and Lemma 5.8, we would have $(a \sqsupset^{\mathfrak{A}} b) \in \mathrm{Fg}^{\mathfrak{A}}_{C}(\varnothing) = \{a \sqsupset^{\mathfrak{A}} a\}$, in which case we would get $(a \sqsupset^{\mathfrak{A}} b) = (a \sqsupset^{\mathfrak{A}} a)$, and so, by (2.4), we would eventually get $(a \uplus^{\mathfrak{A}}_{\sqsupset} b) = ((a \sqsupset^{\mathfrak{A}} b) \sqsupset^{\mathfrak{A}} b) = ((a \sqsupset^{\mathfrak{A}} a) \sqsupset^{\mathfrak{A}} b) = b$. Therefore, by Corollaries 3.16(i)⇒(ii), 3.31 and Lemma 3.24, there is some $\uplus^{\mathfrak{A}}_{\sqsupset}$-disjunctive $G \in \mathrm{Fi}_{C}(\mathfrak{A})$ such that $b \notin G \ni a$, in which case $\mathcal{D} \triangleq \langle \mathfrak{A}, G\rangle \in \mathrm{Mod}(C)$ is finite and $\uplus_{\sqsupset}$-disjunctive, and so, by Corollary 3.20 and Remark 2.8(ii), there are some $\mathcal{B} \in \mathsf{M}$ and some $h \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B}) \subseteq \hom(\mathfrak{A}, \mathfrak{B})$, in which case, as $b \notin G = D^{\mathcal{D}} \ni a$, we have $h(a) \in D^{\mathcal{B}} \not\ni h(b)$, and so we get $\chi^{\mathcal{B}}(h(a)) = 1 \neq 0 = \chi^{\mathcal{B}}(h(b))$. □

5.3. **Common consequences.** The effective procedure of verifying the self-extensionality of an n-valued implicative/"both disjunctive and conjunctive" logic, where $n \in \omega$, resulted from Theorem 5.6/5.10 has the computational complexity n^n that is quite acceptable for (3|4)-valued logics. And what is more, it provides a quite useful heuristic tool of doing it, manual applications of which (suppressing the factor n^n at all) are presented below. First, we have:

Corollary 5.11. *Let $n \in (\omega \setminus 3)$, $\mathcal{A}$ a finite hereditarily simple $\sqsupset$-implicative/"both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive" Σ-matrix and C the logic of $\mathcal{A}$. Suppose every non-singular endomorphism of $\mathfrak{A}$ is diagonal. Then, the logic of $\mathcal{A}$ is not self-extensional.*

Proof. By contradiction. For suppose C is self-extensional. Then, as $n \in (\omega \setminus 3)$, $\mathcal{A}$ is either false- or truth-non-singular, in which case $\chi^{\mathcal{A}}$ is not injective, and so there are some distinct $a, b \in A$ such that $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(b)$. On the other hand, by Theorem 5.6/5.10, there is some $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$, in which case h is not singular, and so $h = \Delta_A$. Hence, $\chi^{\mathcal{A}}(a) = \chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b)) = \chi^{\mathcal{A}}(b)$. This contradiction completes the argument. $\square$

5.3.1. *Self-extensionality versus algebraizability.* We start from proving the following "implicative" analogue of Lemma 11 of [22] being interesting in its own right within the context of Universal Algebra:

Lemma 5.12. *Let $\mathcal{A}$ be an $\sqsupset$-implicative Σ-matrix with [finite] unary unitary equality determinant Υ. Suppose $\mathfrak{A}$ is an $\sqsupset$-implicative inner semi-lattice. Then, $\mho^{\sqsupset}_{\Upsilon} \triangleq \{((\gamma(x_i) \sqsupset \gamma(x_{1-i})) \uplus_{\sqsupset} (\delta(x_{2+j}) \sqsupset \delta(x_{2+1-j}))) \approx (x_0 \sqsupset x_0) \mid i, j \in 2, \gamma, \delta \in \Upsilon\}$ is a [finite] disjunctive system for $\mathfrak{A}$.*

Proof. Consider any $\bar{a} \in A^4$. Let $h \in \hom(\mathfrak{Fm}^4_\Sigma, \mathfrak{A})$ extend $[x_i/a_i]_{i \in 4}$.

First, assume $(a_0 = a_1)|(a_2 = a_3)$. Then, for each $(\gamma|\delta) \in \Upsilon$ and every $(i|j) \in 2$, $(\gamma|\delta)^{\mathfrak{A}}(a_{i|(2+j)}) = (\gamma|\delta)^{\mathfrak{A}}(a_{(1-i)|(2+1-j)})$, in which case $((\gamma|\delta)^{\mathfrak{A}}(a_{i|(2+j)}) \sqsupset^{\mathfrak{A}} (\gamma|\delta)^{\mathfrak{A}}(a_{(1-i)|(2+1-j)})) = \flat^{\mathfrak{A}}_{\uplus_{\sqsupset}}$, and so, for each $(\delta|\gamma) \in \Upsilon$ and every $(j|i) \in 2$, $((\gamma^{\mathfrak{A}}(a_i) \sqsupset^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_{1-i})) \uplus^{\mathfrak{A}}_{\sqsupset} (\delta^{\mathfrak{A}}(a_{2+j}) \sqsupset \delta^{\mathfrak{A}}(a_{2+1-j}))) = \flat^{\mathfrak{A}}_{\uplus_{\sqsupset}} = (a_0 \sqsupset^{\mathfrak{A}} a_0)$. Thus, $\mathfrak{A} \models (\bigwedge \mho^{\sqsupset}_{\Upsilon})[h]$.

Conversely, assume both $a_0 \neq a_1$ and $a_2 \neq a_3$. Then, there are some $\gamma, \delta \in \Upsilon$ and some $i, j \in 2$ such that both $\gamma^{\mathfrak{A}}(a_i) \in D^{\mathcal{A}} \not\ni \gamma^{\mathfrak{A}}(a_{1-i})$ and $\delta^{\mathfrak{A}}(a_{2+j}) \in D^{\mathcal{A}} \not\ni \delta^{\mathfrak{A}}(a_{2+1-j})$, in which case, by the $\sqsupset$-implicativity of $\mathcal{A}$, $(\gamma^{\mathfrak{A}}(a_i) \sqsupset^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_{1-i})) \notin D^{\mathcal{A}} \not\ni (\delta^{\mathfrak{A}}(a_{2+j}) \sqsupset \delta^{\mathfrak{A}}(a_{2+1-j}))$, and so, by the $\uplus_{\sqsupset}$-disjunctivity of $\mathcal{A}$, $((\gamma^{\mathfrak{A}}(a_i) \sqsupset^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_{1-i})) \uplus^{\mathfrak{A}}_{\sqsupset} (\delta^{\mathfrak{A}}(a_{2+j}) \sqsupset \delta^{\mathfrak{A}}(a_{2+1-j}))) \notin D^{\mathcal{A}}$. On the other hand, by the $\sqsupset$-implicativity of $\mathcal{A}$, $(a_0 \sqsupset^{\mathfrak{A}} a_0) \in D^{\mathcal{A}}$. Hence, $((\gamma^{\mathfrak{A}}(a_i) \sqsupset^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_{1-i})) \uplus^{\mathfrak{A}}_{\sqsupset} (\delta^{\mathfrak{A}}(a_{2+j}) \sqsupset \delta^{\mathfrak{A}}(a_{2+1-j}))) \neq (a_0 \sqsupset^{\mathfrak{A}} a_0)$. Thus, $\mathfrak{A} \not\models (\bigwedge \mho^{\sqsupset}_{\Upsilon})[h]$. $\square$

According to [22], given any $m, n \in \omega$, a *(Σ-)equational $\vdash^m_n$-(sequent)definition for* a Σ-matrix $\mathcal{A}$ is any $\Omega \in \wp_\omega(\mathrm{Eq}^{m+n}_\Sigma)$ such that, for all $\bar{a} \in A^m$ and all $\bar{b} \in A^n$, it holds that $(((\mathrm{img}\, a) \subseteq D^{\mathcal{A}}) \Rightarrow (((\mathrm{img}\, b) \cap D^{\mathcal{A}}) \neq \varnothing)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge \Omega)[x_i/a_i; x_{m+j}/b_j]_{i \in m; j \in n})$. (Equational $\vdash^{0/1}_1$-definitions are also referred to as *equational "truth definitions"/implications*, respectively/, according to Appendix A of [24].) Some kinds of equational sequent definitions are actually equivalent for implicative matrices, in view of the following compound immediate observation:

Remark 5.13. Given a[n $\sqsupset$-implicative] Σ-matrix $\mathcal{A}$, (i[-v]) does [resp., do] hold, where:

(i) given any equational $\vdash^2_2$-definition Ω for $\mathcal{A}$, $\Omega[x_{(2 \cdot i)+j}/x_i]_{i,j \in 2}$ is an equational implication for $\mathcal{A}$ (cf. Theorems 10 and 12(ii)$\Rightarrow$(iii) of [24]);
(ii) given any equational implication Ω for $\mathcal{A}$, $\Omega[x_0/(x_0 \sqsupset x_0), x_1/x_0]$ is an equational truth definition for $\mathcal{A}$;
(iii) given any equational truth definition Ω for $\mathcal{A}$, the following hold:
  a) $\Omega[x_0/(x_0 \sqsupset x_1)]$ is an equational implication for $\mathcal{A}$;
  b) $\Omega[x_0/(x_0 \sqsupset (x_1 \sqsupset (x_2 \uplus_{\sqsupset} x_3)))]$ is an equational $\vdash^2_2$-definition for $\mathcal{A}$;
(iv) given any unary [binary] equality determinant ε (in particular, $\varepsilon = \varepsilon_\Upsilon$, where Υ is a [unary] unitary equality determinant) for $\mathcal{A}$, $\{\phi \sqsupset \psi \mid (\phi \vdash \psi) \in \varepsilon\}$ is an axiomatic [binary] equality determinant for $\mathcal{A}$;

(v) in case $\mathcal{A}$ is truth-singular, $\{x_0 \approx (x_0 \sqsupset x_0\}$ is an equational truth definition for it. $\Box$

In this way, taking Theorems 10, 12(i)⇔(ii) and 13 of [22] as well as Remark 5.13 into account, a "both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive"/$\sqsupset$-implicative consistent truth-non-empty finite Σ-matrix $\mathcal{M}$ with unary unitary equality determinant has an equational implication iff a multi-conclusion two-side sequent calculus $\widetilde{\mathcal{S}}^{(k,l)}_{\mathcal{M},\mathcal{T}}$ (cf. [21] as well as the paragraph -2 on p. 294 of [22] for more detail)/" (or the equivalent — in the sense of [18] — logic of $\mathcal{M}$)" is algebraizable — in the sense of [18]. In this connection, by Lemma 9 and Theorem[s] 10 [and 14(ii)⇒(i)] of [22] [as well as Lemma 5.12/"11 of [22]"], we have

Lemma 5.14 (cf. Theorem[s] 14 [and 15] of [22] [for the "lattice conjunctive disjunctive" case])**.** *Let $\mathcal{A}$ be a finite consistent truth-non-empty [$\sqsupset$-implicative/"both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive"] Σ-matrix with unary unitary equality determinant. [Suppose $\mathfrak{A}$ is an/a "$\sqsupset$-implicative inner semi-lattice"/$(\overline{\wedge},\underline{\vee})$-lattice, respectively.] Then, $\mathcal{A}$ has an equational implication [if and] only if every non-singular partial endomorphism of $\mathfrak{A}$ is diagonal.*

As a consequence, by Theorem 3.11(ii)⇒(i), Corollary 5.11 and Lemma 5.14, we immediately get the following universal negative result:

Corollary 5.15. *Let $n \in (\omega \setminus 3)$, $\mathcal{A}$ an n-valued consistent truth-non-empty $\sqsupset$-implicative/"both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive" Σ-matrix with unary unitary equality determinant and C the logic of $\mathcal{A}$. Suppose $\mathcal{A}$ has an equational implication. Then, C is not self-extensional.*

The converse does not, generally speaking, hold — even in the "lattice conjunctive disjunctive" case (cf. Example 6.23), though does hold within the framework of three-valued paraconsistent/paracomplete logics with subclassical negation as well as "lattice conjunction and disjunction"|"implicative inner semi-lattice implication" (cf. Corollary 6.121|6.131, respectively). In view of Theorem 10 and Lemma 8 of [22], Example 5.2 and the self-extensionality of inferentially inconsistent logics, the reservations "$n \in (\omega \setminus 3)$" and "$n$-valued consistent truth-non-empty" cannot be omitted in the formulation of Corollary 5.15.

Example 5.16 (Łukasiewicz' finitely-valued logics; cf. [9])**.** Let $n \in (\omega \setminus 3)$, $\Sigma \triangleq (\Sigma_{+,\sim} \cup \{\supset\})$ with binary $\supset$ (implication) and $\mathcal{A}$ the Σ-matrix with $(\mathfrak{A}{\restriction}\Sigma_+) \triangleq \mathfrak{D}_n$, $D^{\mathcal{A}} \triangleq \{1\}$, $\sim^{\mathfrak{A}} \triangleq (1-a)$ and $(a \supset^{\mathfrak{A}} b) \triangleq \min(1, 1-a+b)$, for all $a, b \in A$, in which case $\mathcal{A}$ is both consistent, truth-non-empty, $\wedge$-conjunctive and $\underline{\vee}$-disjunctive as well as, by Example 7 of [22], is implicative, and so, by Remark 5.13(v),(iii)**a)**, has an equational implication (cf. Example 7 of [22]) and, by Example 3 of [21], a unary unitary equality determinant. Hence, by Corollary 5.15, the logic of $\mathcal{A}$ is not self-extensional. $\Box$

Example 5.17. In view of Remarks 1 and 2 of [22], Lemma 5.14 and Corollaries 5.11 and 5.15, arbitrary three-valued expansions of both the *logic of paradox LP* [14] and Kleene's three-valued logic K_3 [7] are not self-extensional, because the former has the equational implication $(x_0 \wedge (x_1 \vee \sim x_1)) \approx (x_0 \wedge x_1)$, discovered in [17], while the latter has the same underlying algebra. Likewise, in view of "Proposition 5.7 of [24]"/"both Lemma 4.1 of [15] and Remark 5.13(iii)**a)**" as well as Corollary 5.15, arbitrary three-valued expansions of P^1/HZ [26]/[6] are not self-extensional, for they have an equational implication/"truth definition", respectively. $\Box$

Another generic applications of our universal elaboration are discussed in the next section.

6. Applications and examples

6.1. **Four-valued expansions of Belnap's four-valued logic.** Here, it is supposed that $\Sigma \supseteq \Sigma_{+,\sim[,01]}$. Fix a Σ-matrix $\mathcal{A}$ with $(\mathfrak{A}{\restriction}\Sigma_{+,\sim[,01]}) \triangleq \mathfrak{DM}_{4[,01]}$ and $D^{\mathcal{A}} \triangleq (2^2 \cap \pi_0^{-1}[\{1\}])$, Then, both $\mathcal{A}$ and $\partial(\mathcal{A}) \triangleq \langle \mathfrak{A}, 2^2 \cap \pi_1^{-1}[\{1\}]\rangle$ are both $\wedge$-conjunctive and $\vee$-disjunctive, while $\{x_0, \sim x_0\}$ is a unary unitary equality determinant for them (cf. Example 2 of [21]), so they as well as their submatrices are hereditarily simple (cf. Theorem 3.11(ii)$\Rightarrow$(i)), while:

$$(\theta^{\mathcal{A}} \cap \theta^{\partial(\mathcal{A})}) \;=\; \Delta_A, \tag{6.1}$$

$$D^{\partial(\mathcal{A})} \;=\; (\sim^{\mathfrak{A}})^{-1}[A \setminus D^{\mathcal{A}}]. \tag{6.2}$$

Let C be the logic of $\mathcal{A}$. Then, as $\mathcal{DM}_{4[,01]} \triangleq (\mathcal{A}{\restriction}\Sigma_{+,\sim[,01]})$ defines [the bounded version/expansion of] Belnap's four-valued logic $B_{4[,01]}$ [3] (cf. [16]), C is a four-valued expansion of $B_{4[,01]}$. This exhaust all four-valued expansions of $B_{4[,01]}$, $\mathcal{A}$ being uniquely determined by C, as we show below, marking the framework of the present subsection:

Lemma 6.1. *Any $\Sigma_{+,\sim[,01]}$-matrix $\mathcal{B}$ defines $B_{4[,01]}$ and is four-valued iff it is isomorphic to $\mathcal{DM}_{4[,01]}$, in which case it is simple.*

Proof. The "if" part is by (2.23) and the fact that $|2^2| = 4$. Conversely, assume $B_{4[,01]}$ is defined by $\mathcal{B}$, while this is four-valued. Then, by (2.23) and Remark 2.8[(iv)], $\mathcal{D} \triangleq (\mathcal{B}/\theta)$, where $\theta \triangleq \eth(\mathcal{B})$, is a simple $\Sigma_{+,\sim[,01]}$-matrix defining $B_{4[,01]}$. Hence, by Theorem 3.9, $\mathfrak{D}$ and $\mathfrak{DM}_{4[,01]}$ generate the same (intrinsic) variety (of $B_{4[,01]}$), in which case they satisfy same identities, and so the former is a [bounded] De Morgan lattice, for the latter is so. In particular,

$$((x_0 \wedge \sim x_0) \wedge (x_1 \vee \sim x_1)) \approx (x_0 \wedge \sim x_0), \tag{6.3}$$

not being true in the latter under $[x_i/\langle i, 1-i\rangle]_{i\in 2}$, is not true in the former, in which case $\mathfrak{D}{\restriction}\Sigma_+$ is not a chain, and so there are some $a, b \in D$ such that $D \ni (c|d) \triangleq (a(\wedge|\vee)^{\mathfrak{D}} b) \notin \{a, b\}$. Then, $a \neq b$, in which case $c \neq d$, and so $D = \{a, b, c, d\}$, for $|D| \leqslant |B| = 4$. Therefore, $|D| = 4 \nleqslant 3$, in which case θ is diagonal, and so ν_θ is an isomorphism from $\mathcal{B}$ onto $\mathcal{D}$. Hence, $c|d$ is a zero$|$unit of $\mathfrak{D}{\restriction}\Sigma_+$, in which case $[(c|d) = (\bot|\top)^{\mathfrak{D}}$, while], by (2.6)|(2.7), $\sim^{\mathfrak{D}}(c|d) = (d|c)$, and so, by (2.5), $\sim^{\mathfrak{D}}(a/b) \notin \{c, d\}$. On the other hand, if $\sim^{\mathfrak{A}}(a/b)$ was equal to b/a, then, by (2.5), $\sim^{\mathfrak{A}}(b/a)$ was equal to a/b, in which case $e(\wedge|\vee)^{\mathfrak{D}} \sim^{\mathfrak{D}} e$ would be equal to $c|d$, for all $e \in D$, and so (6.3) would be true in $\mathfrak{D}$. Thus, $\sim^{\mathfrak{D}}(a/b) = (a/b)$. And what is more, $\mathcal{D}$ is both consistent, truth-non-empty and $\wedge$-conjunctive, for $\mathcal{DM}_{4[,01]}$ is so, that is, $B_{4[,01]}$ is both inferentially consistent and $\wedge$-conjunctive. Hence, $c \notin D^{\mathcal{D}} \ni d$, in which case $\{a, b\} \nsubseteq D^{\mathcal{D}}$, and so $(\{a, b\} \cap D^{\mathcal{D}}) \neq \varnothing$, for, otherwise, $D^{\mathcal{D}}$ would be equal to $\{d\}$, in which case $\mathcal{D}$ would be non-$\sim$-paraconsistent, and so would be $B_{4[,01]}$, contrary to the fact that (2.16) is not true in $\mathcal{DM}_{4[,01]}$ under $[x_i/\langle 1-i, i\rangle]_{i\in 2}$. Therefore, $D^{\mathcal{D}} = \{d, e\}$, for some $e \in \{a, b\}$, in which case the mapping $g : 2^2 \to D$, given by:

$$\begin{aligned} g(11) &\triangleq d, \\ g(00) &\triangleq c, \\ g(10) &\triangleq e, \\ g(01) &\triangleq f, \end{aligned}$$

where f is the unique element of $\{a, b\} \setminus \{e\}$, is an isomorphism from $\mathcal{DM}_{4[,01]}$ onto $\mathcal{D}$, and so $g^{-1} \circ \nu_\theta$ is that from $\mathcal{B}$ onto $\mathcal{DM}_{4[,01]}$. Finally, the simplicity of the latter and Remark 2.8[(iii)] complete the argument. $\square$

Theorem 6.2. *Any four-valued Σ-expansion C' of $B_{4[,01]}$ is defined by a unique Σ-expansion of $\mathcal{DM}_{4[,01]}$.*

Proof. Let $\mathcal{A}'$ be a four-valued Σ-matrix defining C'. Then, $\mathcal{A}'{\restriction}\Sigma_{+,\sim[,01]}$ is a four-valued $\Sigma_{+,\sim[,01]}$-matrix defining $B_{4[,01]}$, in which case, by Lemma 6.1, there is some isomorphism e from $\mathcal{A}'{\restriction}\Sigma_{+,\sim[,01]}$ onto $\mathcal{DM}_{4[,01]}$, and so e is an isomorphism from $\mathcal{A}'$ onto the Σ-expansion $\mathcal{A}'' \triangleq \langle e[\mathfrak{A}'], 2^2 \cap \pi_0^{-1}[\{1\}]\rangle$ of $\mathcal{DM}_{4[,01]}$. Hence, by (2.23), C' is defined by $\mathcal{A}''$, being both finite and $\veebar$-disjunctive as well as having a unary unitary equality determinant. Finally, let $\mathcal{A}'''$ be any more Σ-expansion of $\mathcal{DM}_{4[,01]}$ defining C', in which case it is a $\vee$-disjunctive model of C', and so, by Theorem 3.23, there is some $h \in \hom_{\mathrm{S}}(\mathcal{A}''', \mathcal{A}'')$. Then, $h \in \hom_{\mathrm{S}}(\mathcal{DM}_{4[,01]}, \mathcal{DM}_{4[,01]})$, in which case, by Lemma 3.12, h is diagonal, and so $\mathcal{A}''' = \mathcal{A}''$, as required. □

Given any $i \in 2$, put $DM_{3,i} \triangleq (2^2 \setminus \{\langle i, 1-i\rangle\})$. Then, in case this forms a subalgebra of $\mathfrak{A}$ (such is the case, when, e.g., $\Sigma = \Sigma_{\sim,+[,01]}$), we set $(\mathcal{A}/\mathcal{DM})_{3,i/[,01]} \triangleq ((\mathcal{A}/\mathcal{DM})_{/4[,01]}{\restriction}DM_{3,i})$, the logic $(C/B)_{3,i/[,01]}$ of which is a both $\vee$-disjunctive and $\wedge$-conjunctive (for its defining matrix is so; cf. Remark 2.9(ii)) as well as inferentially consistent (for its defining matrix is both consistent and truth-non-empty) unitary three-valued both extension of $(C/B)_{4/[,01]}$, in view of (2.23), and a three-valued expansion of [the bounded version/expansion $LP_{01}|K_{3,01}$ of] "the *logic of paradox*"|"Kleene's three-valued logic" $LP|K_3$ [14]|[7], defined by $\mathcal{DM}_{3,i[,01]}$, whenever $i = (0|1)$, in which case it is $\vee$-disjunctive as well as $\sim$-paraconsistent|$(\vee,\sim)$-paracomplete, and so is not $\sim$-classical, in view of Remark 2.9(i)**d)**,(ii).

Let $\mu : 2^2 \to 2^2, \langle i, j\rangle \mapsto \langle j, i\rangle$ and $\sqsubseteq \triangleq \{\langle ij, kl\rangle \in (2^2)^2 \mid i \leqslant k, l \leqslant j\}$, those n-ary operations on 2^2, where $n \in \omega$, which "commute with μ"/"are monotonic with respect to $\sqsubseteq$", being said to be *specular/regular*, respectively. Then, $\mathfrak{A}$ is said to be *specular/regular*, whenever its primary operations are so, in which case secondary ones are so as well. (Clearly, $\mathfrak{DM}_{4[,01]}$ is both specular and regular.) Then:

$$D^{\partial(\mathcal{A})} = \mu^{-1}[D^{\mathcal{A}}]. \tag{6.4}$$

Theorem 6.3. *The following are equivalent:*

(i) *C is self-extensional;*
(ii) *$\mathfrak{A}$ is specular;*
(iii) *$\partial(\mathcal{A})$ is isomorphic to $\mathcal{A}$;*
(iv) *C is defined by $\partial(\mathcal{A})$;*
(v) *$\partial(\mathcal{A}) \in \mathrm{Mod}(C)$;*
(vi) *C has PWC with respect to $\sim$.*

Proof. First, assume (i) holds. Then, by Theorem 5.6, there is some $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(11)) \neq \chi^{\mathcal{A}}(h(10))$, in which case h is not singular, and so $B \triangleq (\mathrm{img}\, h)$ forms a non-one-element subalgebra of $\mathfrak{A}$. Hence $\Delta_2 \subseteq B$, in which case $\mathfrak{A}[{\restriction}B]$ is a $(\wedge,\vee)$-lattice with zero/unit $\langle 0/1, 0/1\rangle$, and so, by Lemma 2.3, $(h{\restriction}\Delta_2)$ is diagonal. Therefore, $h(10) \notin D^{\mathcal{A}}$, for $h(11) = (11) \in D^{\mathcal{A}}$. On the other hand, for all $a \in A$, it holds that $(\sim^{\mathfrak{A}} a = a) \Leftrightarrow (a \notin \Delta_2)$. Therefore, $h(10) = (01)$. Moreover, if $h(01)$ was equal to 01 too, then we would have $(00) = h(00) = h((10) \wedge^{\mathfrak{A}} (01)) = ((01) \wedge^{\mathfrak{A}} (01)) = (01)$. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = \mu$, so (ii) holds.

Next, (ii)⇒(iii) is by (6.4) and the bijectivity of $\mu : A \to A$, while (iii)⇒(iv) is by (2.23), whereas (v) is a particular case of (iv). Further, (i)⇒(vi) is by:

Claim 6.4. *Any self-extensional extension C' of C has PWC with respect to $\sim$.*

Proof. In that case, C' is $\wedge$-conjunctive and satisfies (2.8) with $i = 1$, for C is and does so. Consider any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and any $\psi \in C'(\phi)$, in which case both $\sim(\phi \wedge \psi) \equiv_C (\sim\phi \vee \sim\psi)$, in view of (2.6), true in $\mathfrak{A}$, and Lemma 3.7, and $(\phi \wedge \psi) \equiv_{C'} \phi$, in view the $\wedge$-conjunctivity of C', and so, by (2.8) with $i = 1$ and the self-extensionality of C', $\sim\phi \equiv_{C'} (\sim\phi \vee \sim\psi) \in C'(\sim\psi)$, as required. $\Box$

Now, assume (vi) holds. Consider any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$, any $\psi \in C(\phi)$, in which case $\sim\phi \in C(\sim\psi)$, and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h(\phi) \in D^{\partial(\mathcal{A})}$, in which case, by (6.2), $h(\sim\phi) \notin D^{\mathcal{A}}$, and so $h(\sim\psi) \notin D^{\mathcal{A}}$, that is, $h(\psi) \in D^{\partial(\mathcal{A})}$. Thus, $\partial(\mathcal{A})$, being both truth-non-empty and $\bar{\wedge}$-conjunctive, is a $(2 \setminus 1)$-model of C, and so, by Lemma 5.3, (v) holds.

Finally, (v)$\Rightarrow$(i) is by (6.1) and Theorem 5.1(vi)$\Rightarrow$(i) with $\mathsf{S} = \{\mathcal{A}, \partial(\mathcal{A})\}$. $\Box$

This positively covers $B_{4[,01]}$ as regular instances. And what is more, in case $\Sigma = \Sigma_{\simeq,+[,01]} \triangleq (\Sigma_{\sim,+[,01]} \cup \{\neg\})$ with unary $\neg$ (classical — viz., Boolean — negation) and $\neg^{\mathfrak{A}}\langle i, j\rangle \triangleq \langle 1-i, 1-j\rangle$, Theorem 6.3 equally covers the logic $CB_{4[,01]} \triangleq C$ of the $(\neg x_0 \vee x_1)$-implicative $\mathcal{DMB}_{4[,01]} \triangleq \mathcal{A}$ with non-regular — because of $\neg^{\mathfrak{A}}$ — underlying algebra, introduced in [19]. Below, we disclose a *unique* (up to term-wise definitional equivalence) status of these three self-extensional instances.

Lemma 6.5. *Suppose $\mathfrak{A}$ is specular. Then, Δ_2 forms a subalgebra of $\mathfrak{A}$. In particular, C is $\sim$-subclassical, whenever it is self-extensional.*

Proof. By contradiction. For suppose there are some $f \in \Sigma$ of arity $n \in \omega$ and some $\bar{a} \in \Delta_2^n$ such that $f^{\mathfrak{A}}(\bar{a}) \notin \Delta_2$. Then, $f^{\mathfrak{A}}(\bar{a}) = f^{\mathfrak{A}}(\mu \circ \bar{a}) = \mu(f^{\mathfrak{A}}(\bar{a})) \neq f^{\mathfrak{A}}(\bar{a})$. This contradiction, Theorem 6.3(i)$\Rightarrow$(ii) and (2.23) complete the argument. $\Box$

Corollary 6.6. *Suppose C is self-extensional. Then, the following are equivalent:*

(i) *C has a theorem;*
(ii) *$\top^{\mathfrak{DM}_{4,01}}$ is term-wise definable in $\mathfrak{A}$;*
(iii) *$\bot^{\mathfrak{DM}_{4,01}}$ is term-wise definable in $\mathfrak{A}$;*
(iv) *$\{01\}$ does not form a subalgebra of $\mathfrak{A}$;*
(v) *$\{10\}$ does not form a subalgebra of $\mathfrak{A}$.*

Proof. Then, by Theorem 6.3(i)$\Rightarrow$(ii), $\mu \in \hom(\mathfrak{A}, \mathfrak{A})$. First, (i,iv) are particular cases of (ii), for $(01) \neq \top^{\mathfrak{DM}_{4,01}} = (11) \in D^{\mathcal{A}}$. Next, (ii)$\Leftrightarrow$(iii) is by the equalities $\sim^{\mathfrak{A}}(\bot^{\mathfrak{DM}_{4,01}}/\top^{\mathfrak{DM}_{4,01}}) = (\top^{\mathfrak{DM}_{4,01}}/\bot^{\mathfrak{DM}_{4,01}})$. Likewise, (iv)$\Leftrightarrow$(v) is by the equalities $\mu[\{01/10\}] = \{10/01\}$. Further, (i)$\Rightarrow$(ii) is by Lemmas 5.7 and 6.5. Finally, assume (iv) holds. Then, there is some $\varphi \in \mathrm{Fm}^1_{\Sigma}$ such that $\varphi^{\mathfrak{A}}(01) \neq (01)$, in which case, by the injectivity of μ, we have $(10) = \mu(01) \neq \mu(\varphi^{\mathfrak{A}}(01)) = \varphi^{\mathfrak{A}}(\mu(01)) = \varphi^{\mathfrak{A}}(10)$, and so, by Lemma 6.5, we get $(x_0 \vee (\varphi \vee \sim\varphi)) \in C(\varnothing)$. Thus, (i) holds. $\Box$

Corollary 6.7. *Suppose C is self-extensional, and $\mathcal{A}$ is $\sqsupset$-implicative. Then, $\neg^{\mathfrak{DMB}_4}$ is term-wise definable in $\mathfrak{A}$.*

Proof. Then, by (2.12), true in $\mathcal{A}$, and Corollary 6.6(i)$\Rightarrow$(iii), $\bot^{\mathfrak{DM}_{4,01}} \notin D^{\mathcal{A}}$ is term-wise definable in $\mathfrak{A}$ by some $\tau \in \mathrm{Fm}^1_{\Sigma}$, and so $\mathcal{A}$ is $-$-negative, where $-x_0 \triangleq (x_0 \sqsupset \tau)$. Moreover, by Theorem 6.3, $\mathfrak{A}$ is specular, in which case, by Lemma 6.5, Δ_2 forms a subalgebra of $\mathfrak{A}$, and so $(-^{\mathfrak{A}} \restriction \Delta_2) = (\neg^{\mathfrak{DMB}_4} \restriction \Delta_2)$. On the other hand, if $-^{\mathfrak{A}}(10) \notin D^{\mathcal{A}}$ was equal to 00, then, as $(01) \notin D^{\mathcal{A}}$, we would have $D^{\mathcal{A}} \ni -^{\mathfrak{A}}(01) = -^{\mathfrak{A}}(\mu(10)) = \mu(-^{\mathfrak{A}}(10)) = \mu(00) = (00) \notin D^{\mathcal{A}}$. Therefore, $-^{\mathfrak{A}}(10) = (01)$, in which case $(10) = \mu(01) = \mu(-^{\mathfrak{A}}(10)) = -^{\mathfrak{A}}\mu(10) = -^{\mathfrak{A}}(01)$, and so $-^{\mathfrak{A}} = \neg^{\mathfrak{DMB}_4}$. $\Box$

6.1.1. *Specular functional completeness.* As usual, *Boolean algebras* are supposed to be of the signature $\Sigma^{-} \triangleq (\Sigma_{\simeq,+,01} \setminus \{\sim\})$, the ordinary one over 2 being denoted by $\mathfrak{B}_2$.

Lemma 6.8. *Let $n \in \omega$ and $f : 2^n \to 2$. [Suppose f is monotonic with respect to $\leqslant$ (and $f(n \times \{i\}) = i$, for each $i \in 2$, in which case $n > 0$).] Then, there is some $\vartheta \in \mathrm{Fm}^n_{\Sigma^{-}[\setminus\{\neg(,\bot,\top)\}]}$ such that $g = \vartheta^{\mathfrak{B}_2}$.*

Proof. Then, by the functional completeness of $\mathfrak{B}_2$, there is some $\vartheta \in \mathrm{Fm}^n_{\Sigma^-}$ such that $g = \vartheta^{\mathfrak{B}_2}(\notin \{2^n \times \{i\} \mid i \in 2\})$, in which case, without loss of generality, one can assume that $\vartheta = (\wedge\langle\bar{\varphi}, \top\rangle)$, where, for each $m \in \ell \triangleq (\mathrm{dom}\,\bar{\varphi}) \in (\omega(\backslash 1))$, $\varphi_m = (\vee\langle(\neg \circ \bar{\phi}_m) * \bar{\psi}_m, \bot\rangle)$, for some $\bar{\phi}_m \in V_n^{k_m}$, some $\bar{\psi}_m \in V_n^{l_m}$ and some $k_m, l_m \in \omega$ such that $((\mathrm{img}\,\bar{\phi}_m) \cap (\mathrm{img}\,\bar{\psi}_m)) = \varnothing$. [Set $\zeta \triangleq (\wedge\langle\bar{\eta}, \top\rangle)$, where, for each $m \in (\mathrm{dom}\,\bar{\eta}) \triangleq \ell$, $\eta_m \triangleq (\vee\langle\bar{\psi}_m, \bot\rangle)$. Consider any $\bar{a} \in A^n$ and the following exhaustive cases:

(1) $g(\bar{a}) = 0$,
in which case we have $\zeta^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} \leqslant \vartheta^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} = 0$, and so we get $\zeta^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} = 0$.

(2) $g(\bar{a}) = 1$,
in which case, for every $m \in \ell$, as $\bar{a} \leqslant \bar{b}_m \triangleq ((\bar{a}{\restriction}(n \setminus N_m)) \cup (N_m \times \{1\})) \in A^n$, where $N_m \triangleq \{j \in n \mid x_j \in (\mathrm{img}\,\bar{\phi}_m)\}$, by the monotonicity of g w.r.t. $\leqslant$, we have $1 = g(\bar{a}) \leqslant g(\bar{b}_m) = \vartheta^{\mathfrak{B}_2}[x_j/b_{m,j}]_{j\in n} \leqslant \varphi_m^{\mathfrak{B}_2}[x_j/b_{m,j}]_{j\in n} = \eta_m^{\mathfrak{B}_2}[x_j/a_j]_{j\in n}$, and so we get $\zeta^{\mathfrak{B}_2}[x_j/a_j]_{j\in n} = 1$.

Thus, $g = \zeta^{\mathfrak{B}_2}$. (And what is more, since, in that case, $\ell > 0$ and $l_m > 0$, for each $m \in \ell$, we also have $g = \xi^{\mathfrak{B}_2}$, where $\xi \triangleq (\wedge\bar{\upsilon})$, whereas, for each $m \in (\mathrm{dom}\,\bar{\upsilon}) \triangleq \ell$, $\upsilon_m \triangleq (\vee\bar{\psi}_m)$.)] This completes the argument. □

Theorem 6.9. *Let $\Sigma = \Sigma_{\simeq,+,01}$, $n \in (\omega(\backslash 1))$ and $f : A^n \to A$. Then, f is specular [and regular (as well as $f(n \times \{a\}) = a$, for all $a \in (A \setminus \Delta_A)$)] iff there is some $\tau \in \mathrm{Fm}^n_{\Sigma[\backslash\{\neg(,\bot,\top)\}]}$ such that $f = \tau^{\mathfrak{A}}$.*

Proof. The "if" part is immediate. Conversely, assume f is specular [and regular (as well as $f(n \times \{a\}) = a$, for all $a \in (A \setminus \Delta_A)$)]. Then,

$$g : 2^{2\cdot n} \to 2, \bar{a} \mapsto \pi_0(f(\langle\langle a_{2\cdot j}, 1 - a_{(2\cdot j)+1}\rangle\rangle_{j\in n}))$$

[is monotonic w.r.t. $\leqslant$ (and $g(n \times \{i\}) = i$, for each $i \in 2$)]. Therefore, by Lemma 6.8, there is some $\vartheta \in \mathrm{Fm}^{2\cdot n}_{\Sigma^-[\backslash\{\neg(,\bot,\top)\}]}$ such that $g = \vartheta^{\mathfrak{B}_2}$. Put

$$\tau \triangleq (\vartheta[x_{2\cdot j}/x_j, x_{(2\cdot j)+1}/{\sim}x_j]_{j\in n}) \in \mathrm{Fm}^n_{\Sigma[\backslash\{\neg(,\bot,\top)\}]}.$$

Consider any $\bar{c} \in A^n$. Then, since, for each $i \in 2$, we have $\pi_i \in \hom(\mathfrak{A}{\restriction}\Sigma^-, \mathfrak{B}_2)$, we get $\pi_0(\tau^{\mathfrak{A}}[x_j/c_j]_{j\in n}) = \vartheta^{\mathfrak{B}_2}[x_{2\cdot j}/\pi_0(c_j), x_{(2\cdot j)+1}/(1 - \pi_1(c_j))]_{j\in n} = \pi_0(f(\bar{c}))$ and, likewise, as f is specular, $\pi_1(\tau^{\mathfrak{A}}[x_j/c_j]_{j\in n}) = \vartheta^{\mathfrak{B}_2}[x_{2\cdot j}/\pi_1(c_j), x_{(2\cdot j)+1}/(1 - \pi_0(c_j))]_{j\in n} = \pi_0(f(\mu \circ \bar{c})) = \pi_0(\mu(f(\bar{c}))) = \pi_1(f(\bar{c}))$, as required. □

In this way, by Theorems 6.2, 6.3 and 6.9, $CB_{4[,01]}$ is the most expansive (up to term-wise definitional equivalence) self-extensional four-valued expansion of B_4. And what is more, combining Theorems 6.3 and 6.9 with Corollaries 6.6 and 6.7, we eventually get:

Corollary 6.10. *C is self-extensional, while $\mathcal{A}$ is implicative/"both $\mathfrak{A}$ is regular and C is [not] purely-inferential", iff C is term-wise definitionally equivalent to $CB_4/B_{4[,01]}$, respectively.*

6.1.2. *The structural completion of the bounded expansion.* Here, it is supposed that $\Sigma \triangleq \Sigma_{\sim,+,01}$, in which case $\mathcal{A} = \mathcal{DM}_{4,01}$, and so $C = B_{4,01}$.

Given any bounded De Morgan lattice $\mathfrak{B}$, we have both the truth-singular Σ-matrix $(\mathfrak{B} + \top) \triangleq \langle\mathfrak{B}, \{\top^{\mathfrak{B}}\}\rangle$ and the bounded De Morgan lattice $(\mathfrak{B} \dotplus 2) \triangleq ((\mathfrak{B} \times \mathfrak{K}_{3,01}){\restriction}((B \times \{\frac{1}{2}\}) \cup \{\langle\bot^{\mathfrak{B}}, 0\rangle, \langle\bot^{\mathfrak{B}}, 1\rangle\}))$, for $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{K}_3$. Then, set $\mathfrak{DM}_{10,01} \triangleq ((\mathfrak{DM}_{4,01} \times \mathfrak{K}_{2,01}) \dotplus 2)$.

Theorem 6.11. *$K_{3,01}$ is the structural completion of C.*

Proof. Then, $\mathfrak{A} = \mathfrak{DM}_{4,01}$ is specular, in which case, by Theorem 6.3(ii)⇒(v), $\partial(\mathcal{A})$ is a model of C, and so is $\mathcal{D} \triangleq (\mathfrak{A} + \top)$, for $\{\top^{\mathfrak{A}}\} = \{\langle 1,1\rangle\} = (D^{\mathcal{A}} \cap D^{\partial(\mathcal{A})})$, that is, the logic C' of $\mathcal{D}$ is an extension of C. Moreover, $\Delta_A \in \hom^{\mathrm{S}}(\mathcal{D},\mathcal{A})$, in which case, by (2.24), $T \triangleq C'(\varnothing) = C(\varnothing)$, and so the structural completion of C is that of C'. We find the latter with using Theorem 4.1. Then, $\mathcal{D}$ is generated by $f(\mathcal{D}) \triangleq (A \setminus \Delta_2)$ of cardinality 2. Put:

$$\begin{aligned}
\theta &\triangleq \theta^2_{\mathfrak{A}},\\
a &\triangleq (x_0 \wedge {\sim}x_0),\\
b &\triangleq (x_1 \vee {\sim}x_1),\\
c &\triangleq ((a \vee b) \wedge {\sim}a),\\
d &\triangleq (b \wedge ({\sim}a \vee {\sim}b)),\\
h &\triangleq [x_i/\langle i, 1-i\rangle]_{i\in 2} \in \hom(\mathfrak{Fm}^2_\Sigma, \mathfrak{A}),\\
g_j &\triangleq [x_i/\tfrac{1}{j-1}]_{i\in 2} \in \hom(\mathfrak{Fm}^2_\Sigma, \mathfrak{K}_{j,01}), \quad (j \in (4\setminus 2))\\
e &\triangleq ((h \times f_2) \times f_3) \in \hom(\mathfrak{Fm}^2_\Sigma, (\mathfrak{DM}_{4,01} \times \mathfrak{K}_{2,01}) \times \mathfrak{K}_{3,01})
\end{aligned}$$

and $\mathfrak{S}$ the subalgebra of $\mathfrak{Fm}^2_\Sigma$ generated by $H_2 \triangleq \{c,d\}$, in which case $B \triangleq (S/\theta)$ forms a subalgebra of $\mathfrak{F}^2_{\mathfrak{A}}$, and so $\nu'_\theta \triangleq (\nu_\theta \restriction S)$ is a strict surjective homomorphism from $\mathcal{S} \triangleq \langle \mathfrak{S}, S \cap T\rangle$ onto $\mathcal{B} \triangleq (\mathcal{F}^2_{\mathcal{D}} \restriction B)$. Moreover, by (2.22), for each $\varphi \in D^{\mathcal{S}}$ and every $g' \in \hom(\mathfrak{Fm}^2_\Sigma, \mathfrak{A})$, $g'(\varphi) = \top^{\mathfrak{A}} = g'(\top)$, in which case $\varphi\ \theta\ \top \in D^{\mathcal{D}}$, and so $\mathcal{B} = (\mathfrak{B} + \top)$. Therefore, as $h(c|d) = \langle 0|1, 1|0\rangle$, we have $h[S] = A$, for $\mathfrak{A}$ is thus generated by $h[H_2]$, in which case, by the Homomorphism Theorem, $((h\restriction S)\circ {\nu'}^{-1}_\theta) \in \hom(\mathfrak{B},\mathfrak{A}) \subseteq \hom(\mathcal{B},\mathcal{D})$ is surjective, and so $\mathcal{D}$ is a surjective homomorphic image of $\mathcal{B}$. Thus, the structural completion C'' of $C[']$ is defined by $\mathcal{B}$. And what is more, $(e_3 \circ g_3), (e_2 \circ g_2) \in \hom(\mathfrak{Fm}^2_\Sigma, \mathfrak{A})$, so $\theta \subseteq ((\ker h) \cap (\ker(e_2 \circ g_2)) \cap (\ker(e_3 \circ g_3))) = ((\ker h) \cap (\ker g_2) \cap (\ker g_3)) = (\ker e)$, for both e_2 and e_3 are injective. On the other hand, as $e(c|d) = \langle\langle 0|1, 1|0\rangle, 1, \frac{1}{2}\rangle \in DM_{10,01}$, we have $e[S] = DM_{10,01}$, for $\mathfrak{DM}_{10,01}$ is thus generated by $e[H_2]$. Hence, by the Homomorphism Theorem, $e' \triangleq ((e\restriction S) \circ {\nu'}^{-1}_\theta) \in \hom(\mathfrak{B}, \mathfrak{DM}_{10,01}) \subseteq \hom(\mathcal{B}, \mathfrak{DM}_{10,01} + \top)$ is surjective. Furthermore, as $\mathfrak{S}$ is generated by H_2, $\mathfrak{B}$ is generated by $\nu'_\theta[H_2] = \nu_\theta[H_2]$, for $H_2 \subseteq S$. Put $H_{10} \triangleq (\{\bot,\top\} \cup H_2 \cup {\sim}[H_2] \cup \{{\sim}^k(c \diamond d) | k \in 2, \diamond \in \Sigma_+\})$. Then, since $\mathfrak{F}^2_{\mathcal{D}} = \mathfrak{F}^2_{\mathfrak{A}}$, being isomorphic to a subalgebra of a direct power of $\mathfrak{A}$, is a bounded De Morgan lattice, it is routine checking that $\nu_\theta[H_{10}]$ forms a subalgebra of it. Moreover, $H_2 \subseteq H_{10}$, in which case $B \subseteq \nu_\theta[H_{10}]$, and so, if e' was not injective, then we would have $10 = |DM_{10,01}| = |e'[B]| < |B| \leqslant |\nu_\theta[H_{10}]| \leqslant |H_{10}| = 10$. Hence, e' is injective, in which case it is an isomorphism from $\mathcal{B}$ onto $\mathcal{E} \triangleq (\mathfrak{DM}_{10,01} + \top)$, and so, by (2.23), C'' is defined by $\mathcal{E}$. Finally, $(e_3 \circ (\pi_2 \restriction DM_{10,01})) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}, \mathcal{DM}_{3,1,01})$. Thus, by (2.23), $C'' = K_{3,01}$, as required. □

6.1.3. *No-more-than-three-valued extensions.*

Lemma 6.12. *Let $n \in (4 \setminus 1)$. Then, any n-valued model/extension of C is $\vee$-disjunctive.*

Proof. Let $\mathcal{B}$ be an n-valued model of C, in which case, by (2.23) and Remark 2.8[(iv)], $\mathcal{D} \triangleq (\mathcal{B}/\partial(\mathcal{B}))$, is an m-valued simple model of C, where $m \leqslant n \leqslant 3$, and so, by Corollary 3.8, $\mathfrak{D} \in \mathbf{V}(\mathfrak{A})$. Therefore, $\mathfrak{D}\restriction\Sigma_+$, being an m-element lattice, for $\mathfrak{A}\restriction\Sigma_+$ is a lattice, is a chain. Hence, $\mathcal{D}$, being $\wedge$-conjunctive, for C is so, is $\vee$-disjunctive, and so is $\mathcal{B}$, by Remark 2.9(ii), as required. □

Corollary 6.13. *Let $\mathcal{B}$ be a consistent truth-non-empty non-${\sim}$-negative three-valued model of C and C' the logic of $\mathcal{B}$. Then, there is some $i \in 2$ such that $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B}$ is isomorphic to $\mathcal{A}_{3,i}$, and so $C' = C_{3,i}$.*

Proof. Then, by Lemma 6.12, $\mathcal{B}$ is $\vee$-disjunctive. Hence, by Theorem 3.23, there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, while h is a strict surjective homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A} \upharpoonright D)$. Therefore, if h was not injective, then $\mathcal{D}$ would be either one-valued, in which case it would be either inconsistent or truth-empty, and so would be $\mathcal{B}$, or two-valued, in which case D would be equal to Δ_2, and so, by Remark 2.9(ii), $\mathcal{B}$ would be $\sim$-negative, for $\mathcal{D}$ would be so. Thus, h is injective, in which case $|D| = 3$, and so $D = DM_{3,i}$, for some $i \in 2$. In this way, (2.23) completes the argument. □

Likewise, we have:

Corollary 6.14. *Let $\mathcal{B}$ be a consistent truth-non-empty two-valued model of C and C' the logic of $\mathcal{B}$. Then, Δ_2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B}$ is isomorphic to $\mathcal{A} \upharpoonright \Delta_2$, in which case it is $\sim$-classical, and so is C'.*

Proof. Then, by Lemma 6.12, $\mathcal{B}$ is $\vee$-disjunctive. Hence, by Theorem 3.23, there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, while h is a strict surjective homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A} \upharpoonright D)$. Therefore, if h was not injective, then $\mathcal{D}$ would be one-valued, in which case it would be either inconsistent or truth-empty, and so would be $\mathcal{B}$. Thus, h is injective, in which case $|D| = 2$, and so $D = \Delta_2$. In this way, Remark 2.9(ii) completes the argument. □

And what is more, we also have:

Lemma 6.15. *Let $\mathcal{B}$ be a $\sim$-negative model of C and C' the logic of $\mathcal{B}$. Then, Δ_2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{B}$ is a strict surjective homomorphic counter-image of $\mathcal{A} \upharpoonright \Delta_2$, an so C' is $\sim$-classical.*

Proof. Then, by the following auxiliary observation, $\mathcal{B}$ is $\vee$-disjunctive:

Claim 6.16. *Any $\sim$-negative $\mathcal{B} \in \mathrm{Mod}(C)$ is $\vee$-disjunctive.*

Proof. Then, by Remark 2.9(i)**a)**, $\mathcal{B}$, being $\wedge$-conjunctive, for C is so, is $\wedge^{\sim}$-disjunctive. On the other hand, as (2.5) and (2.7) are true in $\mathfrak{A}$, so is $(x_0 \vee x_1) \approx (x_0 \wedge^{\sim} x_1)$, in which case, by Lemma 3.7, $(x_0 \vee x_1) \equiv^{\omega}_{C} (x_0 \wedge^{\sim} x_1)$, and so $((a \vee^{\mathfrak{B}} b) \in D^{\mathcal{B}}) \Leftrightarrow ((a(\wedge^{\sim})^{\mathfrak{B}} b) \in D^{\mathcal{B}})$, for all $a, b \in B$. Thus, $\mathcal{B}$, being $\wedge^{\sim}$-disjunctive, is equally $\vee$-disjunctive, as required. □

Hence, by Theorem 3.23, there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$, in which case $D \triangleq (\operatorname{img} h)$ forms a subalgebra of $\mathfrak{A}$, while h is a strict surjective homomorphism from $\mathcal{B}$ onto $\mathcal{D} \triangleq (\mathcal{A} \upharpoonright D)$, and so, by Remark 2.9(ii), $\mathcal{D}$ is $\sim$-negative, for $\mathcal{B}$ is so. Therefore, $D = \Delta_2$. Finally, (2.23) completes the argument. □

By Corollary 6.14, Lemma 6.15 and (2.23), we immediately have:

Theorem 6.17. *The following are equivalent:*

(i) *C is $\sim$-subclassical;*
(ii) *C has a consistent truth-non-empty two-valued model;*
(iii) *C has a $\sim$-negative model;*
(iv) *Δ_2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A} \upharpoonright \Delta_2$ is a $\sim$-classical model of C isomorphic to any consistent truth-non-empty two-valued (in particular, $\sim$-classical) model of C and being a strict surjective homomorphic image of any $\sim$-negative model of C, and so defines a unique inferentially consistent two-valued (in particular, $\sim$-classical) extension of C.*

Likewise, Examples 5.2, 5.17, Corollary 6.13, Lemma 6.15 and the self-extensionality of inferentially inconsistent logics then immediately yield:

Theorem 6.18. *Let C' be a three-valued extension of C. Then, the following are equivalent:*

(i) *C' is self-extensional;*
(ii) *C' is either inferentially inconsistent or $\sim$-classical;*
(iii) *for each $i \in 2$, if $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$, then $C' \neq C_{3,i}$.*

In general, since $\mathcal{DM}_4{\restriction}\{01\}$ is the only truth-empty submatrix of $\mathcal{DM}_4$, by Corollaries 3.21, 6.13, Theprem 6.17 and (2.23), we also have:

Theorem 6.19. *Let M be a non-empty class of consistent no-more-than-three-valued models of C, C' the logic of M, $n \in (4 \setminus 1)$ and $\mathsf{M}^{(*)\{,\sim|\not\sim\}}_{[n]\langle,0/1\rangle}$ the class of all (truth-non-empty) [n-valued] $\{\sim$-negative$|$non-$\sim$-negative$\}$ $\langle$false-/truth-singular$\rangle$ members of M. Then, C' is defined by $\{\mathcal{A}{\restriction}\{01\} \mid (\mathsf{M} \setminus \mathsf{M}^*) \neq \varnothing = \mathsf{M}^{*,\not\sim}_{3,1}\} \cup \{\mathcal{A}{\restriction}\Delta_2 \mid (\bigcup_{i\in 2} \mathsf{M}^{*,\not\sim}_{3,i}) = \varnothing \neq (\mathsf{M}^{\sim} \cup \mathsf{M}^*_2)\} \cup \bigcup_{i\in 2}\{\mathcal{A}_{3,i} \mid \mathsf{M}^{*,\not\sim}_{3,i} \neq \varnothing\}$.*

In view of Theorem 6.18, any inferentially consistent non-$\sim$-classical *unitary* three-valued extension of C' is not self-extensional. Then, taking (2.20), Theorem 6.19, Remark 2.7 and Example 5.2 into account, for analyzing the "non-unitary" case it suffices to restrict our consideration by the following "double" one.

6.1.3.1. Double three-valued extension. Here, it is supposed that, for each $i \in 2$, $DM_{3,i}$ forms a subalgebra of $\mathfrak{A}$, in which case, by (2.23), the logic $(C/B)_{3/[,01]}$ of $\{(\mathcal{A}/\mathcal{DM})_{3,0/[,01]}, (\mathcal{A}/\mathcal{DM})_{3,1/[,01]}\}$ is the $\vee$-disjunctive both $\sim$-paraconsistent (for $(\mathcal{A}/\mathcal{DM})_{3,0/[,01]}$ is so) — in particular, non-$\sim$-classical — and $(\vee,\sim)$-paracomplete (for $(\mathcal{A}/\mathcal{DM})_{3,1/[,01]}$ is so) proper extension of $C/B_{4[,01]}$ satisfying $\{x_0, \sim x_0\} \vdash (x_1 \vee \sim x_1)$, for this is not true in $\mathcal{A}/\mathcal{DM}_{4[,01]}$ under $[x_i/\langle 1-i, i\rangle]_{i\in 2}$, and so $\Delta_2 = (DM_{3,0} \cap DM_{3,1})$ forms a subalgebra of $\mathfrak{A}_{[3,0]}$, in which case $C_{[3]}$ is $\sim$-subclassical, in view of (2.23). Moreover, set $\partial(\mathcal{A}_{3,i}) \triangleq (\partial(\mathcal{A}){\restriction}DM_{3,i})$.

Theorem 6.20. *The following are equivalent:*

(i) *C_3 is self-extensional;*
(ii) *for each $i \in 2$, $(\mu{\restriction}DM_{3,i}) \in \hom(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i})$;*
(iii) *for some $i \in 2$, $(\mu{\restriction}DM_{3,i}) \in \hom(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i})$;*
(iv) *for each $i \in 2$, C_3 is defined by $\{\mathcal{A}_{3,i}, \partial(\mathcal{A}_{3,i})\}$;*
(v) *for some $i \in 2$, C_3 is defined by $\{\mathcal{A}_{3,i}, \partial(\mathcal{A}_{3,i})\}$;*
(vi) *for each $i \in 2$, $\partial(\mathcal{A}_{3,i}) \in \mathrm{Mod}(C_3)$;*
(vii) *for some $i \in 2$, $\partial(\mathcal{A}_{3,i}) \in \mathrm{Mod}(C_3)$;*
(viii) *$\mathfrak{A}_{3,0}$ and $\mathfrak{A}_{3,1}$ are isomorphic;*
(ix) *C_3 has PWC with respect to $\sim$;*
(x) *$\mathfrak{A}$ has a non-diagonal non-singular partial endomorphism.*

Proof. First, assume (i) holds. Consider any $i \in 2$. Then, as $DM_{3,i} \ni a \triangleq \langle 1-i, i\rangle \neq b \triangleq \langle 1-i, 1-i\rangle \in \Delta_2 \subseteq DM_{3,i}$, by Theorem 5.6, there are some $j \in 2$, some $h \in \hom(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,j})$ such that $\chi^{\mathcal{A}_{3,j}}(h(a)) \neq \chi^{\mathcal{A}_{3,j}}(h(b))$, in which case h is not singular, and so $B \triangleq (\mathrm{img}\, h)$ forms a non-one-element subalgebra of $\mathfrak{A}_{3,j}$. Therefore, $\Delta_2 \subseteq B$. Hence, $\mathfrak{A}_{3,i[-i+j]}[{\restriction}B]$ is a $(\wedge,\vee)$-lattice with zero/unit $\langle 0/1, 0/1\rangle$, in which case, by Lemma 2.3, $(h{\restriction}\Delta_2)$ is diagonal, and so $h(b) = b \in D^{\mathcal{A}_j}$. On the other hand, for all $c \in A$, it holds that $(\sim^{\mathfrak{A}} c = c) \Leftrightarrow (c \notin \Delta_2)$. Therefore, as $a \notin \Delta_2$, $h(a) \notin \Delta_2$, in which case $B \neq \Delta_2$, and so $B = DM_{3,j}$. Hence, if j was equal to i, we would have $h(a) = a$, in which case we would get $\chi^{\mathcal{A}_{3,j}}(h(a)) = \chi^{\mathcal{A}_{3,j}}(a) = (1-i) = \chi^{\mathcal{A}_{3,j}}(b) = \chi^{\mathcal{A}_{3,j}}(h(b))$, and so $j = (1-i)$, in which case $h(a) = \mu(a)$. Thus, $\hom(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i}) \ni h = (\mu{\restriction}DM_{3,i})$, and so (ii) holds.

Next, (iii/v/vii) is a particular case of (ii/iv/vi), respectively, while (viii) is a particular case of (iii). Likewise, (vi/vii) is a particular case of (iv/v), while (ii/iii)$\Rightarrow$(iv/v) is by (2.23) and (6.4).

Further, assume (vii) holds. Then, as no false-/truth-singular Σ-matrix is isomorphic to any one not being so, while $\partial(\mathcal{A}_{3,i})$ is false-/truth-singular iff $\mathcal{A}_{3,i}$ is not so, by Remarks 2.8(ii), 2.9(ii) and Corollary 3.20, we conclude that $\partial(\mathcal{A}_{3,i})$ is isomorphic to $\mathcal{A}_{3,1-i}$, and so (2.23) yields (v).

Now, assume (viii) holds. Let e be any isomorphism from $\mathfrak{A}_{3,0}$ onto $\mathfrak{A}_{3,1}$. Then, since these are both $(\wedge,\vee)$-lattices with zero/unit $\langle 0/1, 0/1\rangle$, by Lemma 2.3, $e{\upharpoonright}\Delta_2$ is diagonal. Moreover, for all $c \in A$, it holds that $({\sim}^{\mathfrak{A}}c = c) \Leftrightarrow (c \notin \Delta_2)$. Therefore, $e(10) = (01)$, in which case $\hom(\mathfrak{A}_{3,0}, \mathfrak{A}_{3,1}) \ni e = (\mu{\upharpoonright}DM_{3,0})$, and so (iii) with $i = 0$ holds.

Furthermore, (v)$\Rightarrow$(i) is by Theorem 5.1(vi)$\Rightarrow$(i) with $\mathsf{S} = \mathsf{M} = \{\mathcal{A}_{3,i}, \partial(\mathcal{A}_{3,i})\}$ and (6.1), while (i)$\Rightarrow$(ix) is by Claim 6.4.

Conversely, assume (ix) holds. Consider any $i \in 2$, any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$, any $\psi \in C_3(\phi)$, in which case ${\sim}\phi \in C_3({\sim}\psi)$, and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A}_{3,i})$ such that $h(\phi) \in D^{\partial(\mathcal{A}_{3,i})}$, in which case, by (6.2), $h({\sim}\phi) \notin D^{\mathcal{A}_{3,i}}$, and so $h({\sim}\psi) \notin D^{\mathcal{A}_{3,i}}$, that is, $h(\psi) \in D^{\partial(\mathcal{A}_{3,i})}$. Thus, $\partial(\mathcal{A}_{3,i})$ is a $(2 \setminus 1)$-model of C. Moreover, by Remark 2.9(ii), it is $\bar{\wedge}$-conjunctive, for $\partial(\mathcal{A})$ is so, and so, by Lemma 5.3, (vi) holds.

Finally, (x) is a particular case of (iii). Conversely, assume (x) holds. Then, there are some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and some non-diagonal non-singular $h \in \hom(\mathfrak{B}, \mathfrak{A})$, in which case $D \triangleq (\mathrm{img}\, h)$ forms a non-one-element subalgebra of $\mathfrak{A}$, and so does $B = (\mathrm{dom}\, h)$. Hence, $\Delta_2 \subseteq (B \cap D)$. Therefore, both $\mathfrak{B}$ and $\mathfrak{D}$ are $(\wedge,\vee)$-lattices with zero/unit $\langle 0/1, 0/1\rangle$, in which case, as $h \in \hom(\mathfrak{B}, \mathfrak{D})$ is surjective, by Lemma 2.3, $h{\upharpoonright}\Delta_2$ is diagonal, and so there is some $i \in 2$ such that $DM_{3,i} \subseteq B$, while $h(\langle 1-i, i\rangle) \neq \langle 1-i, i\rangle$. On the other hand, for all $a \in A$, it holds that $({\sim}^{\mathfrak{A}}a = a) \Leftrightarrow (a \notin \Delta_2)$, in which case ${\sim}^{\mathfrak{A}}h(\langle 1-i, i\rangle) = h({\sim}^{\mathfrak{A}}\langle 1-i, i\rangle) = h(\langle 1-i, i\rangle)$, and so $h(\langle 1-i, i\rangle) = \langle i, 1-i\rangle$. In this way, $\hom(\mathfrak{A}_{3,i}, \mathfrak{A}) \ni (h{\upharpoonright}DM_{3,i}) = (\mu{\upharpoonright}DM_{3,i})$, in which case $(\mu{\upharpoonright}DM_{3,i}) \in \hom(\mathfrak{A}_{3,i}, \mathfrak{A}_{3,1-i})$, and so (iii) holds, as required. □

First, by Lemma 5.14 and Theorem 6.20(i)$\Leftrightarrow$(x), we immediately have:

Corollary 6.21. *C_3 is self-extensional iff $\mathcal{A}$ has no equational implication.*

Then, by Corollaries 5.15 and 6.21, we also have:

Corollary 6.22. *C_3 is self-extensional, whenever C is so.*

On the other hand, the converse does not hold, as it follows from:

Example 6.23 (cf. Example 11 of [22])**.** Let $\Sigma \triangleq (\Sigma_{\sim,+[,01]} \cup \{\amalg\})$ with binary $\amalg$ and $\amalg^{\mathfrak{A}} \triangleq ((\vee^{\mathfrak{A}}{\upharpoonright}(DM^2_{3,0} \cup DM^2_{3,1})) \cup \{\langle\langle 01, 10\rangle, 11\rangle, \langle\langle 10, 01\rangle, 00\rangle\})$. Then, $\mathfrak{A}$ is not specular, while $(\mu{\upharpoonright}DM_{3,0}) \in \hom(\mathfrak{A}_{3,0}, \mathfrak{A}_{3,1})$. Hence, by Theorems 6.3, 6.20 and Corollary 6.21, C_3 is self-extensional, while C is not so, whereas $\mathcal{A}$ has no equational implication. □

6.2. **Three-valued logics with subclassical negation.** A Σ-matrix $\mathcal{A}$ is said to be ${\sim}$-*super-classical*, if $\mathcal{A}{\upharpoonright}\{\sim\}$ has a ${\sim}$-classical submatrix, in which case $\mathcal{A}$ is both consistent and truth-non-empty, while, by (2.23), ${\sim}$ is a subclassical negation for the logic of $\mathcal{A}$, and so we have the "if" part of the following preliminary marking the framework of the present subsection:

Theorem 6.24. *Let $\mathcal{A}$ be a Σ-matrix. [Suppose $|A| \leqslant 3$.] Then, ${\sim}$ is a subclassical negation for the logic of $\mathcal{A}$ if[f] $\mathcal{A}$ is ${\sim}$-super-classical.*

Proof. [Assume ${\sim}$ is a subclassical negation for the logic of $\mathcal{A}$. First, by (3.11) with $m = 1$ and $n = 0$, there is some $a \in D^{\mathcal{A}}$ such that ${\sim}^{\mathfrak{A}}a \notin D^{\mathcal{A}}$. Likewise, by (3.11) with $m = 0$ and $n = 1$, there is some $b \in (A \setminus D^{\mathcal{A}})$ such that ${\sim}^{\mathfrak{A}}b \in D^{\mathcal{A}}$, in which case $a \neq b$, and so $|A| \neq 1$. Then, if $|A| = 2$, we have $A = \{a, b\}$, in which case $\mathcal{A}$ is ${\sim}$-classical, and so ${\sim}$-super-classical. Now, assume $|A| = 3$.

Claim 6.25. *Let $\mathcal{A}$ be a three-valued Σ-matrix, $\bar{a} \in A^2$ and $i \in 2$. Suppose $\sim$ is a subclassical negation for the logic of $\mathcal{A}$ and, for each $j \in 2$, $(a_j \in D^{\mathcal{A}}) \Leftrightarrow (\sim^{\mathfrak{A}} a_j \notin D^{\mathcal{A}}) \Leftrightarrow (a_{1-j} \notin D^{\mathcal{A}})$. Then, either $\sim^{\mathfrak{A}} a_i = a_{1-i}$ or $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_i$.*

Proof. By contradiction. For suppose both $\sim^{\mathfrak{A}} a_i \neq a_{1-i}$ and $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i \neq a_i$. Then, in case $a_i \in / \notin D^{\mathcal{A}}$, as $|A| = 3$, we have both $(D^{\mathcal{A}}/(A \setminus D^{\mathcal{A}})) = \{a_i\}$, in which case $\sim^{\mathfrak{A}} a_{1-i} = a_i$, and $((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}}) = \{a_{1-i}, \sim^{\mathfrak{A}} a_i\}$, respectively. Consider the following exhaustive cases:

- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_{1-i}$.
  Then, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = a_i$. This contradicts to (3.11) with $(n/m) = 0$ and $(m/n) = 3$, respectively.
- $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a_i = \sim^{\mathfrak{A}} a_i$.
  Then, for each $c \in ((A \setminus D^{\mathcal{A}})/D^{\mathcal{A}})$, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} c = \sim^{\mathfrak{A}} a_i \notin / \in D^{\mathcal{A}}$. This contradicts to (3.11) with $(n/m) = 3$ and $(m/n) = 0$, respectively.

Thus, in any case, we come to a contradiction, as required. □

Finally, consider the following exhaustive cases:

- both $\sim^{\mathfrak{A}} a = b$ and $\sim^{\mathfrak{A}} b = a$.
  Then, $\{a, b\}$ forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $(\mathcal{A}\restriction\{\sim\})\restriction\{a, b\}$ being a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$, as required.
- $\sim^{\mathfrak{A}} a \neq b$.
  Then, by Claim 6.25, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a = a$, in which case $\{a, \sim^{\mathfrak{A}} a\}$ forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $(\mathcal{A}\restriction\{\sim\})\restriction\{a, \sim^{\mathfrak{A}} a\}$ being a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$, as required.
- $\sim^{\mathfrak{A}} b \neq a$.
  Then, by Claim 6.25, $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} b = b$, in which case $\{b, \sim^{\mathfrak{A}} b\}$ forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $(\mathcal{A}\restriction\{\sim\})\restriction\{b, \sim^{\mathfrak{A}} b\}$ being a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$, as required.] □

The following counterexample shows that the optional condition $|A| \leqslant 3$ is essential for the optional "only if" part of Theorem 6.24 to hold:

Example 6.26. Let $n \in \omega$ and $\mathcal{A}$ any Σ-matrix with $A \triangleq (n \cup (2 \times 2))$, $D^{\mathcal{A}} \triangleq \{\langle 1, 0\rangle, \langle 1, 1\rangle\}$, $\sim^{\mathfrak{A}}\langle i, j\rangle \triangleq \langle 1-i, (1-i+j) \mod 2\rangle$, for all $i, j \in 2$, and $\sim^{\mathfrak{A}} k \triangleq \langle 1, 0\rangle$, for all $k \in n$. Then, for any subalgebra $\mathfrak{B}$ of $\mathfrak{A}\restriction\{\sim\}$, we have $(2 \times 2) \subseteq B$, in which case $4 \leqslant |B|$, and so $\mathcal{A}$ is not $\sim$-super-classical, for $4 \nleqslant 2$. On the other hand, 2×2 forms a subalgebra of $\mathfrak{A}\restriction\{\sim\}$, $\mathcal{B} \triangleq (\mathcal{A}\restriction\{\sim\})\restriction(2 \times 2)$ being $\sim$-negative, in which case $\chi^{\mathcal{A}}\restriction(2 \times 2)$ is a surjective strict homomorphism from $\mathcal{B}$ onto the $\sim$-classical $\{\sim\}$-matrix $\mathcal{C}$ with $C \triangleq 2$, $D^{\mathcal{C}} \triangleq \{1\}$ and $\sim^{\mathfrak{C}} i \triangleq (1-i)$, for all $i \in 2$, and so, by (2.23), $\sim$ is a subclassical negation for the logic of $\mathcal{A}$. □

Let $\mathcal{A}$ be a three-valued $\sim$-super-classical (in particular, both consistent and truth-non-empty) Σ-matrix and $\mathcal{B}$ a $\sim$-classical submatrix of $\mathcal{A}\restriction\{\sim\}$. Then, as $4 \nleqslant 3$, $\mathcal{A}$ is either false-singular, in which case the unique non-distinguished value $0_{\mathcal{A}}$ of $\mathcal{A}$ is that $0_{\mathcal{B}}$ of $\mathcal{B}$, so $1^{\sim}_{\mathcal{A}} \triangleq \sim^{\mathfrak{A}} 0_{\mathcal{A}} = \sim^{\mathfrak{B}} 0_{\mathcal{B}} = 1_{\mathcal{B}}$, or truth-singular, in which case the unique distinguished value $1_{\mathcal{A}}$ of $\mathcal{A}$ is that $1_{\mathcal{B}}$ of $\mathcal{B}$, so $0^{\sim}_{\mathcal{A}} \triangleq \sim^{\mathfrak{A}} 1_{\mathcal{A}} = \sim^{\mathfrak{B}} 1_{\mathcal{B}} = 0_{\mathcal{B}}$. Thus, in case $\mathcal{A}$ is false-/truth-singular, $B = 2^{\sim}_{\mathcal{A}} \triangleq \{0^{/\sim}_{\mathcal{A}}, 1^{\sim/}_{\mathcal{A}}\}$ is uniquely determined by $\mathcal{A}$ and $\sim$, the unique element of $A \setminus 2^{\sim}_{\mathcal{A}}$ being denoted by $(\frac{1}{2})^{\sim}_{\mathcal{A}}$. (The indexes ${}_{\mathcal{A}}$ and, especially, ${}^{\sim}$ are often omitted, unless any confusion is possible.) Strict homomorphisms from $\mathcal{A}$ to itself retain both 0 and 1, in which case surjective ones retain $\frac{1}{2}$, and so:

$$\hom^{[\mathrm{S}]}_{\mathrm{S}}(\mathcal{A}, \mathcal{A}) \supseteq [=]\{\Delta_A\}, \tag{6.5}$$

the inclusion [not] being allowed to be proper (cf. Example 6.32 below). Then, $\mathcal{A}$ is said to be *canonical*, provided $A = (3 \div 2)$ and $a_{\mathcal{A}} = a$, for all $a \in A$.

Lemma 6.27. *Any isomorphism e between canonical three-valued $\sim$-super-classical Σ-matrices $\mathcal{A}$ and $\mathcal{B}$ is diagonal, in which case $\mathcal{A} = \mathcal{B}$.*

Proof. Then, by Remark 2.9(ii), $\mathcal{A}$ is "false-/-truth-singular"|$\sim$-negative iff $\mathcal{B}$ is so, in which case $D^{\mathcal{A}} = D^{\mathcal{B}}$, and so $\sim^{\mathfrak{A}} \frac{1}{2}$ is equal to $0/1$ iff $\sim^{\mathfrak{B}} \frac{1}{2}$ is so. Moreover, since $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, we have $(\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}) \Leftrightarrow (\mathfrak{A} \models \exists_1(\sim x_0 \approx x_0)) \Leftrightarrow (\mathfrak{B} \models \exists_1(\sim x_0 \approx x_0)) \Leftrightarrow (\sim^{\mathfrak{B}} \frac{1}{2} = \frac{1}{2})$. Hence, $\sim^{\mathfrak{A}} = \sim^{\mathfrak{B}}$. In this way, e is an isomorphism between common three-valued $\sim$-super-classical $\sim$-reducts of $\mathcal{A}$ and $\mathcal{B}$, in which case, by (6.5), e is diagonal, and so $\mathcal{A} = \mathcal{B}$, as required. □

Lemma 6.28. *Any three-valued $\sim$-super-classical Σ-matrix $\mathcal{A}$ is isomorphic to a unique canonical one.*

Proof. Then, the mapping $e : (3 \div 2) \mapsto A, a \mapsto a_{\mathcal{A}}$ is a bijection, in which case it is an isomorphism from the canononical three-valued $\sim$-super-classical Σ-matrix $\langle e^{-1}[\mathfrak{A}], e^{-1}[D^{\mathcal{A}}]\rangle$ onto $\mathcal{A}$. In this way, Lemma 6.27 completes the argument. □

As an immediate consequence of (2.23), Theorem 6.24 and Lemma 6.28, we have:

Corollary 6.29. *Unitary three-valued Σ-logics with subclassical negation $\sim$ are exactly Σ-logics defined by single canonical three-valued $\sim$-super-classical Σ-matrices.*

From now on, unless otherwise specified, C is supposed to be the logic of an arbitrary but fixed canonincal three-valued $\sim$-super-classical Σ-matrix $\mathcal{A}$. (In view of Corollary 6.29, this exhaust all three-valued Σ-logics with subclassical negation $\sim$.) Then, C is $\barwedge$-conjunctive iff $\mathcal{A}$ is so. It appears that such does hold for both disjunctivity and implicativity too, as it ensues from the following two lemmas:

Lemma 6.30. *Let $\mathcal{B}$ be a Σ-matrix and C' the logic of $\mathcal{B}$. Suppose [either] $\mathcal{B}$ is false-singular (in particular, $\sim$-classical) [or both $\mathcal{B}$ is $\sim$-super-classical and $|B| \leqslant 3$]. Then, the following are equivalent:*

(i) *C' is $\veebar$-disjunctive;*
(ii) *$\mathcal{B}$ is $\veebar$-disjunctive;*
(iii) (2.8) *with $i = 0$,* (2.9) *and* (2.10) *[as well as the* Resolution *rule:*

$$\{x_0 \veebar x_1, \sim x_0 \veebar x_1\} \vdash x_1] \tag{6.6}$$

are satisfied in C' (viz., true in $\mathcal{B}$);
(iv) (2.8) *with $i = 0$,* (2.9) *and* (2.10) *[as well as the* Modus ponens *rule for the* material implication *$\sim x_0 \veebar x_1$:*

$$\{x_0, \sim x_0 \veebar x_1\} \vdash x_1] \tag{6.7}$$

are satisfied in C' (viz., true in $\mathcal{B}$).

Proof. First, (ii)$\Rightarrow$(i) is immediate.

Next, assume (i) holds. Then, (2.8) with $i = 0$, (2.9) and (2.10) are immediate. [In addition, suppose $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, and so it is both truth-singular and, therefore, not $\sim$-paraconsistent. Hence, $x_1 \in (C'(x_1) \cap C'(\{x_0, \sim x_0\})) = (C'(x_1) \cap C'(\{x_0 \veebar x_1, \sim x_0\})) = C'(\{x_0 \veebar x_1, \sim x_0 \veebar x_1\})$, so (6.6) is satisfied in C'.] Thus, (iii) holds.

Further, (iv) is a particular case of (iii) [for (6.7) is that of (6.6), in view of (2.8) with $i = 0$].

Finally, assume (iv) holds. Consider any $a, b \in B$. In case $(a/b) \in D^{\mathcal{B}}$, by (2.8) with $i = 0$ /"and (2.9)", we have $(a \veebar^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Now, assume $(\{a, b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.10), we get

$D^{\mathcal{B}} \not\ni (a \veebar^{\mathfrak{B}} a) = (a \veebar^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, whereas (6.7) is true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, while $\sim^{\mathfrak{B}}\sim^{\mathfrak{B}} c = c$. Let d be the unique element of $\{a, b\} \setminus \{c\}$, in which case $\{a, b\} = \{c, d\}$. Then, since $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, we conclude that $(c \veebar^{\mathfrak{B}} d) = (\sim^{\mathfrak{B}}\sim^{\mathfrak{B}} c \veebar^{\mathfrak{B}} d) \notin D^{\mathcal{B}}$, for, otherwise, by (6.7), we would get $d \in D^{\mathcal{B}}$. Hence, by (2.9), we eventually get $(a \veebar^{\mathfrak{B}} b) \notin D^{\mathcal{B}}$.] Thus, (ii) holds, as required. □

Lemma 6.31. *Let $\mathcal{B}$ be a Σ-matrix and C' the logic of $\mathcal{B}$. Suppose [either] $\mathcal{B}$ is false-singular (in particular, $\sim$-classical) [or both $\mathcal{B}$ is $\sim$-super-classical and $|B| \leqslant 3$]. Then, the following [but (i)] are equivalent:*

(i) *C' is weakly $\sqsupset$-implicative;*
(ii) *C' is $\sqsupset$-implicative;*
(iii) *$\mathcal{B}$ is $\sqsupset$-implicative;*
(iv) (2.12)*,* (2.13) *and* (2.11) *[as well as both* (2.15) *and the* Ex Contradictione Quodlibet *axiom:*

$$\sim x_0 \sqsupset (x_0 \sqsupset x_1)] \tag{6.8}$$

are satisfied in C' (viz., true in $\mathcal{B}$).

In particular, any $\sim$-classical/"three-valued $\sim$-paraconsistent" Σ-logic /"with subclassical negation $\sim$" is $\sqsupset$-implicative iff it is weakly so.

Proof. First, (iii)⇒(ii) is immediate, while (i) is a particular case of (ii).

Next, assume (i[i]) holds. Then, (2.12), (2.13) and (2.11) [as well as (2.15)] are immediate. [In addition, suppose $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, and so it is both truth-singular and, therefore, non-$\sim$-paraconsistent, and so is C'. Hence, by Deduction Theorem, (6.8) is satisfied in C'.] Thus, (iv) holds.

Finally, assume (iv) holds. Consider any $a, b \in B$. In case $b \in D^{\mathcal{B}}$, by (2.13) and (2.11), we have $(a \sqsupset^{\mathfrak{B}} b) \in D^{\mathcal{B}}$. Likewise, in case $\{a, a \sqsupset^{\mathfrak{B}} b\} \subseteq D^{\mathcal{B}}$, by (2.11), we have $b \in D^{\mathcal{B}}$. Now, assume $(\{a, b\} \cap D^{\mathcal{B}}) = \varnothing$. Then, in case $a = b$ (in particular, $\mathcal{B}$ is false-singular), by (2.12), we get $D^{\mathcal{B}} \ni (a \sqsupset^{\mathfrak{B}} a) = (a \sqsupset^{\mathfrak{B}} b)$. [Otherwise, $\mathcal{B}$ is not false-singular, in which case it is $\sim$-super-classical, while $|B| \leqslant 3$, whereas both (2.15) and (6.8) and true in $\mathcal{B}$, and so, for some $c \in (B \setminus D^{\mathcal{B}}) = \{a, b\}$, it holds that $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$. Let d be the unique element of $\{a, b\} \setminus \{c\}$, in which case $\{a, b\} = \{c, d\}$. Then, since $\sim^{\mathfrak{B}} c \in D^{\mathcal{B}}$, by (2.11) and (6.8), we conclude that $(c \sqsupset^{\mathfrak{B}} d) \in D^{\mathcal{B}}$. Let us prove, by contradiction, that $(d \sqsupset^{\mathfrak{B}} c) \in D^{\mathcal{B}}$. For suppose $(d \sqsupset^{\mathfrak{B}} c) \notin D^{\mathcal{B}}$, in which case $(d \sqsupset^{\mathfrak{B}} c) = (c/d)$, and so we have $((d \sqsupset^{\mathfrak{B}} c) \sqsupset^{\mathfrak{B}} d) = ((c \sqsupset^{\mathfrak{B}} d)/(d \sqsupset^{\mathfrak{B}} d)) \in D^{\mathcal{B}}/$, by (2.12). Hence, by (2.11) and (2.15), we get $d \in D^{\mathcal{B}}$. This contradiction shows that $(d \sqsupset^{\mathfrak{B}} c) \in D^{\mathcal{B}} \ni (c \sqsupset^{\mathfrak{B}} d)$. In particular, we eventually get $(a \sqsupset^{\mathfrak{B}} b) \in D^{\mathcal{B}}$.] Thus, (iii) holds, as required/", in view of Corollary 6.29". □

Next, we have the *dual* three-valued $\sim$-super-classical Σ-matrix $\partial(\mathcal{A}) \triangleq \langle \mathfrak{A}, \{1\} \cup (\{\frac{1}{2}\} \cap (A \setminus D^{\mathcal{A}}))\rangle$, in which case it is false/truth-singular iff $\mathcal{A}$ is not so, while:

$$(\theta^{\mathcal{A}} \cap \theta^{\partial(\mathcal{A})}) = \Delta_A. \tag{6.9}$$

Likewise, set $\mathcal{A}_{a[+(b)]} \triangleq \langle \mathfrak{A}, \{[\frac{1}{2}(-\frac{1}{2} + b),]a\}\rangle$, where $a[(,b)] \in A$, in which case $(\partial(\mathcal{A})/\mathcal{A}) = \mathcal{A}_{1[+]}$, whenever $\mathcal{A}$ is [not] false-/truth-singular, while:

$$(\theta^{\mathcal{A}_{a[+]}} \cap \theta^{\mathcal{A}_{b[+]}}) = \Delta_A, \tag{6.10}$$

for all distinct $a, b \in A$.

Further, given any $i \in 2$, put $h_i \triangleq (\Delta_2 \cup \{\langle \frac{1}{2}, i\rangle\}) : (3 \div 2) \to 2$, in which case:

$$h_{0/1}^{-1}[D^{\mathcal{A}}] = D^{\partial(\mathcal{A})}, \tag{6.11}$$

whenever $\mathcal{A}$ is false-/truth-singular.

Finally, let $h_{1-} : (3 \div 2) \to (3 \div 2), a \mapsto (1 - a)$, in which case:

$$h_{1-}^{-1}[D^{\mathcal{A}_{i[+]}}] = D^{\mathcal{A}_{(1-i)[+]}}, \tag{6.12}$$

for all $i \in 2$.

Three-valued logics with subclassical negation $\sim$ (even both implicative — in particular, disjunctive — and conjunctive ones) need not, generally speaking, be non-$\sim$-classical, as it ensues from the following elementary example:

Example 6.32. Let $\Sigma \triangleq \Sigma_{+,\sim}$ and $(\mathcal{B}/\mathcal{E})|\mathcal{F}$ the $\wedge$-conjunctive $\vee$-disjunctive $\sim$-negative "false-/truth-singular canonical three-valued $\sim$-super-classical"|$\sim$-classical Σ-matrix with $(((\mathfrak{B}/\mathfrak{E})|\mathfrak{F})\restriction\Sigma_+) \triangleq \mathfrak{D}_{3|2}$. Then, $(\mathcal{B}/\mathcal{E})|\mathcal{F}$ is $\sqsupset_{\vee}^{\sim}$-implicative, in view of Remark 2.9(i)**c**). And what is more, $\chi^{\mathcal{B}/\mathcal{E}} \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}/\mathcal{E}, \mathcal{F})$. Therefore, by (2.23), $\mathcal{B}/\mathcal{E}$ define the same $\sim$-classical Σ-logic of $\mathcal{F}$. On the other hand, $\mathcal{B}$, being false-singular, is not isomorphic to $\mathcal{E}$, not being so. Moreover, $h \triangleq (\Delta_2 \circ \chi^{\mathcal{B}/\mathcal{E}})$ is a non-diagonal (for $h(\frac{1}{2}) = (1/0) \neq \frac{1}{2}$) strict homomorphism from $\mathcal{B}/\mathcal{E}$ to itself, so the "[]"-non-optional inclusion in (6.5) may be proper. □

On the other hand, $\sim$-classical three-valued Σ-logics with subclassical negation $\sim$ are self-extensional, in view of Example 5.2. This makes the characterization to be obtained below especially acute.

Lemma 6.33. *Let $\mathcal{B}$ be a three-valued $\sim$-super-classical Σ-matrix. Then, following are equivalent:*

(i) *$\mathcal{B}$ is a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix;*
(ii) *$\mathcal{B}$ is not simple;*
(iii) *$\mathcal{B}$ is not hereditarily simple;*
(iv) *$\theta^{\mathcal{B}} \in \mathrm{Con}(\mathfrak{B})$.*

Proof. First, (i)⇒(ii) is by Remark 2.8(ii) and the fact that $3 \not\leq 2$. Next, (iii) is a particular case of (ii). The converse is by the fact that any proper submatrix of $\mathcal{B}$, being either one-valued or $\wr$-classical, is simple. Further, (ii)⇒(iv) is by the following claim:

Claim 6.34. *Let $\mathcal{B}$ be a three-valued as well as both consistent and truth-non-empty Σ-matrix. Then, any non-diagonal congruence θ of it is equal to $\theta^{\mathcal{B}}$.*

Proof. First, we have $\theta \subseteq \theta^{\mathcal{B}}$. Conversely, consider any $\bar{a} \in \theta^{\mathcal{B}}$. Then, in case $a_0 = a_1$, we have $\bar{a} \in \Delta_B \subseteq \theta$. Otherwise, take any $\bar{b} \in (\theta \setminus \Delta_B) \neq \varnothing$, in which case $\bar{b} \in \theta^{\mathcal{B}}$, for $\theta \subseteq \theta^{\mathcal{B}}$. Then, as $|B| = 3 \not\geqslant 4$, there are some $i, j \in 2$ such that $a_i = b_j$. Hence, if a_{1-i} was not equal to b_{1-j}, then we would have both $|\{a_i, a_{1-i}, b_{1-j}\}| = 3 = |B|$, in which case we would get $\{a_i, a_{1-i}, b_{1-j}\} = B$, and $\chi^{\mathcal{B}}(b_{1-j}) = \chi^{\mathcal{B}}(b_j) = \chi^{\mathcal{B}}(a_i) = \chi^{\mathcal{B}}(a_{1-i})$, and so $\mathcal{B}$ would be either truth-empty or inconsistent. Therefore, both $a_{1-i} = b_{1-j}$ and $a_i = b_j$. Thus, since θ is symmetric, we eventually get $\bar{a} \in \theta$, for $\bar{b} \in \theta$, as required. □

Finally, assume (iv) holds. Then, $\theta \triangleq \theta^{\mathcal{B}}$, including itself, is a congruence of $\mathcal{B}$, in which case $\nu_\theta \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{B}, \mathcal{B}/\theta)$, while $\mathcal{B}/\theta$ is $\sim$-classical, and so (i) holds. □

Set $h_{+/2} : 2^2 \to (3 \div 2), \langle i, j \rangle \mapsto \frac{i+j}{2}$.

Theorem 6.35. *The following are equivalent:*

(i) *C is $\sim$-classical;*
(ii) *$\mathcal{A}$ is either a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix or a strict surjective homomorphic image of a submatrix of a direct power of a $\sim$-classical Σ-matrix;*

(iii) *either $\mathcal{A}$ is a strict surjective homomorphic counter-image of a $\sim$-classical Σ-matrix or $\mathcal{A}$ is a strict surjective homomorphic image of the direct square of a $\sim$-classical Σ-matrix;*
(iv) *either $\mathcal{A}$ is not simple or both 2 forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}$ is a strict surjective homomorphic image of $(\mathcal{A}{\restriction}2)^2$;*
(v) *either $\theta^{\mathcal{A}} \in \mathrm{Con}(\mathfrak{A})$ or both 2 forms a subalgebra of $\mathfrak{A}$, $\mathcal{A}$ is truth-singular and $h_{+/2} \in \hom((\mathfrak{A}{\restriction}2)^2, \mathfrak{A})$.*

Proof. We use Lemma 6.33 tacitly. First, (ii/iii/iv) is a particular case of (iii/iv/v), respectively. Next, (iv)⇒(i) is by (2.23). Further, (i)⇒(ii) is by Lemma 2.12 and Remark 2.8(ii).

Now, let $\mathcal{B}$ be a $\sim$-classical Σ-matrix, I a set, $\mathcal{D}$ a submatrix of $\mathcal{B}^I$ and $h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{A})$, in which case $\mathcal{D}$ is both consistent and truth-non-empty, for $\mathcal{A}$ is so, and so $I \neq \varnothing$. Take any $a \in D^{\mathcal{B}} \neq \varnothing$. Then, as $\mathcal{B}$ is truth-singular, $D \ni a = (I \times \{1_{\mathcal{B}}\}) \in D^{\mathcal{D}}$, in which case $D \ni b \triangleq \sim^{\mathfrak{D}} a = (I \times \{0_{\mathcal{B}}\}) \notin D^{\mathcal{D}}$, for $I \neq \varnothing$, while $\sim^{\mathfrak{D}} b = a$, and so $E \triangleq \{a, b\}$ forms a subalgebra of $\mathfrak{D}{\restriction}\{\sim\}$, $\mathcal{E} \triangleq ((\mathcal{D}{\restriction}\{\sim\}){\restriction}E)$ being $\sim$-classical with $1_{\mathcal{E}} = a$ and $0_{\mathcal{E}} = b$, and so being $(\mathcal{A}{\restriction}\{\sim\}){\restriction}h[E])$, in view of Remark 2.9(ii). Hence, $h(a/b) = (1/0)$. Therefore, there is some $c \in (D \setminus \{a, b\})$ such that $h(c) = \frac{1}{2}$. In this way, $I \neq J \triangleq \{i \in I \mid \pi_i(c) = 1_{\mathcal{B}}\} \neq \varnothing$. Given any $\bar{a} \in B^2$, set $(a_0 \| a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in B^I$. Then, $D \ni a = (1_{\mathcal{B}} \| 1_{\mathcal{B}})$ and $D \ni b = (0_{\mathcal{B}} \| 0_{\mathcal{B}})$ as well as $D \ni c = (1_{\mathcal{B}} \| 0_{\mathcal{B}})$, in which case $D \ni \sim^{\mathfrak{D}} c = (0_{\mathcal{B}} \| 1_{\mathcal{B}})$, and so $e \triangleq \{\langle\langle x, y\rangle, (x \| y)\rangle \mid x, y \in B\}$ is an embedding of $\mathcal{B}^2$ into $\mathcal{D}$ such that $\{a, b, c\} \subseteq (\mathrm{img}\, e)$. Hence, since $h[\{a, b, c\}] = A$, we conclude that $(h \circ e) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}^2, \mathcal{A})$. Thus, (ii)⇒(iii) holds.

Likewise, let $\mathcal{B}$ be a $\sim$-classical Σ-matrix and $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}^2, \mathcal{A})$. Then, $e' \triangleq (\Delta_B \times \Delta_B)$ is an embedding of $\mathcal{B}$ into $\mathcal{B}^2$, in which case, by Remark 2.8(ii), $g' \triangleq (g \circ e')$ is an embedding of $\mathcal{B}$ into $\mathcal{A}$, and so $E \triangleq (\mathrm{img}\, g')$ forms a two-element subalgebra of $\mathfrak{A}$, g' being an isomorphism from $\mathcal{B}$ onto $\mathcal{E} \triangleq (\mathcal{A}{\restriction}E)$, in which case $h \triangleq ((g'^{-1} \circ (\pi_0{\restriction}E^2)) \times (g'^{-1} \circ (\pi_1{\restriction}E^2)))$ is an isomorphism from $\mathcal{E}^2$ onto $\mathcal{B}^2$. Therefore, as $\mathfrak{A}{\restriction}\{\sim\}$ has no two-element subalgebra other than that with carrier 2, $E = 2$. And what is more, $(g \circ h) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{E}^2, \mathcal{A})$. Thus, (iii)⇒(iv) holds.

Finally, assume (iv) holds, while $\mathcal{A}$ is simple. Then, $\mathcal{A}$ is truth-singular, for $\mathcal{F} \triangleq (\mathcal{A}{\restriction}2)$ is so. Let $f \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F}^2, \mathcal{A})$. Then, $\langle 1, 1\rangle \in D^{\mathcal{F}^2}$, in which case $f(\langle 1, 1\rangle) \in D^{\mathcal{A}}$, and so $f(\langle 1, 1\rangle) = 1$. Hence, $f(\langle 0, 0\rangle) = f(\sim^{\mathfrak{A}^2}\langle 1, 1\rangle) = \sim^{\mathfrak{A}} f(\langle 1, 1\rangle) = \sim^{\mathfrak{A}} 1 = 0$. Moreover, $\sim^{\mathfrak{A}^2}\langle 0/1, 1/0\rangle = \langle 1/0, 0/1\rangle \notin D^{\mathcal{F}^2}$. Hence, $f(\langle 0/1, 1/0\rangle) \notin D^{\mathcal{A}} \not\ni \sim^{\mathfrak{A}} f(\langle 0/1, 1/0\rangle)$. Therefore, $f(\langle 0/1, 1/0\rangle) = \frac{1}{2}$. Thus, $f = h_{+/2}$, so (v) holds. □

Corollary 6.36. *[Providing $\mathcal{A}$ is either false-singular or $\overline{\wedge}$-conjunctive or $\underline{\vee}$-disjunctive] C is $\sim$-classical if[f] $\mathcal{A}$ is not (hereditarily) simple.*

Proof. The "if" part is by Theorem 6.35(iv)⇒(i) (and Lemma 6.33(iii)⇒(ii)). [The converse is proved by contradiction. For suppose C is $\sim$-classical, while $\mathcal{A}$ is simple. Then, by Lemma 6.33(iv)⇒(ii) and Theorem 6.35(i)⇒(v), 2 forms a subalgebra of $\mathfrak{A}$, while $h \triangleq h_{+/2} \in \hom((\mathfrak{A}{\restriction}2)^2, \mathfrak{A})$, whereas $\mathcal{A}$ is truth-singular, in which case it is not false-singular, and so $\overline{\wedge}$-conjunctive|$\underline{\vee}$-disjunctive, and so is $\mathcal{A}{\restriction}2$, in view of Remark 2.9(ii). Hence, $(i(\overline{\wedge}|\underline{\vee})^{\mathfrak{A}} j) = (\min|\max)(i, j)$, for all $i, j \in 2$. Therefore, $\frac{1}{2} = h(01) = h((01)(\overline{\wedge}|\underline{\vee})^{\mathfrak{A}^2}(01)) = (h(01)(\overline{\wedge}|\underline{\vee})^{\mathfrak{A}^2} h(01)) = (\frac{1}{2}(\overline{\wedge}|\underline{\vee})^{\mathfrak{A}^2}\frac{1}{2}) = (h(01)(\overline{\wedge}|\underline{\vee})^{\mathfrak{A}^2} h(10)) = h((01)(\overline{\wedge}|\underline{\vee})^{\mathfrak{A}^2}(10)) = h((00)|(11)) = (0|1)$. This contradiction completes the argument.] □

Generally speaking, the optional stipulation cannot be omitted in the formulation of Corollary 6.36, even if C is *weakly* conjunctive/disjunctive, as it follows from:

Example 6.37. Let $\Sigma \triangleq \{\diamond, \sim\}$ with binary $\diamond$ and $\mathcal{A}$ truth-singular with $(a \diamond^{\mathfrak{A}} b) \triangleq (0/1)$ and $\sim^{\mathfrak{A}} a \triangleq (1-a)$, for all $a, b \in A$. Then, $\mathcal{A}$ is weakly $\diamond$-conjunctive/-disjunctive, respectively, while $\langle 0, \frac{1}{2}\rangle \in \theta^{\mathcal{A}} \not\ni \langle 1, \frac{1}{2}\rangle = \langle \sim^{\mathfrak{A}} 0, \sim^{\mathfrak{A}} \frac{1}{2}\rangle$, in which case $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, and so, by Lemma 6.33(ii)⇒(iv), $\mathcal{A}$ is simple. On the other hand, 2 forms a subalgebra of $\mathfrak{A}$, while $h_{+/2} \in \hom((\mathfrak{A}{\restriction}2)^2, \mathfrak{A})$. Hence, by Theorem 6.35(v)⇒(i), C is $\sim$-classical. □

Theorem 6.38. *Let $\mathcal{B}$ be a [canonical] three-valued $\sim$-super-classical Σ-matrix. Suppose C is defined by $\mathcal{B}$ as well as non-$\sim$-classical. Then, $\mathcal{B}$ is isomorphic [equal] to $\mathcal{A}$.*

Proof. In that case, $\mathcal{A}$ (as well as $\mathcal{B}$) is simple, in view of ((2.23), Remark 2.8[(iii)], Lemma 6.28 and) Theorem 6.35(iv)⇒(i).

Consider the following complementary cases:

- $\mathcal{B}$ is $\sim$-paraconsistent.
  Then, it is false-singular, and so weakly $\sim$-negative. Moreover, any proper submatrix of $\mathcal{B}$ is either $\sim$-classical or one-valued (in which case it is either truth-empty or inconsistent), and so is not $\sim$-paraconsistent. Therefore, by Remark 2.8(ii) and Lemma 6.41, there is an embedding of $\mathcal{A}$ into $\mathcal{B}$, being then an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, because $|A| = 3 \leqslant n$, for no $n \in 3 = |B|$.
- $\mathcal{B}$ (and so $\mathcal{A}$) is not $\sim$-paraconsistent.
  Then, as $\mathcal{B}$ is simple and finite, by Lemma 2.12 and Remark 2.8(ii), there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it and some $g \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$, in which case $\mathcal{D}$ is both truth-non-empty and consistent (in particular, $I \neq \varnothing$), for $\mathcal{B}$ is so. Given any $x \in A$, set $(I : x) \triangleq (I \times \{x\}) \in A^I$. Then, by the following claim, $a \triangleq (I : 1) \in D \ni b \triangleq (I : 0)$:

  Claim 6.39. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{D}$ a subdirect product of it. Suppose $\mathcal{A}$ is weakly conjunctive, whenever it is $\sim$-paraconsistent, and $\mathcal{D}$ is truth-non-empty, otherwise. Then, $\{I \times \{j\} \mid j \in 2\} \subseteq D$.*

  Proof. Consider the following complementary cases:
  - $\mathcal{A}$ is $\sim$-paraconsistent,
    in which case it is false-singular and weakly conjunctive, and so, by Lemma 3.1, $b \triangleq (I \times \{0\}) \in D$.
  - $\mathcal{A}$ is not $\sim$-paraconsistent,
    in which case $\mathcal{D}$ is truth-non-empty. Take any $a \in D^{\mathcal{D}} \neq \varnothing$. Let $b \triangleq \sim^{\mathfrak{D}} a \in D$. Consider any $i \in I$. Then, $\pi_i(a) \in D^{\mathcal{A}}$. Consider the following complementary subcases:
    * $\frac{1}{2} \in D^{\mathcal{A}}$,
      in which case, since $\mathcal{A}$ is not $\sim$-paraconsistent but is consistent, $\pi_i(b) = \sim^{\mathfrak{A}} \pi_i(a) \notin D^{\mathcal{A}}$, and so, as $1 \in D^{\mathcal{A}}$, $\pi_i(b) = 0$.
    * $\frac{1}{2} \notin D^{\mathcal{A}}$,
      in which case, as $0 \notin D^{\mathcal{A}}$, $\pi_i(a) = 1$, and so $\pi_i(b) = \sim^{\mathfrak{A}} \pi_i(a) = 0$.

    In this way, $D \ni b = (I \times \{0\})$.

  Then, $D \ni \sim^{\mathfrak{D}} b = (I \times \{1\})$. □

  Consider the following complementary subcases:
  - 2 does not form a subalgebra of $\mathfrak{A}$,
    in which case there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(1, 0) = \frac{1}{2}$, and so $D \in \varphi^{\mathfrak{D}}(a, b) = (I : \frac{1}{2})$. In this way, as $I \neq \varnothing$, $e \triangleq \{\langle x, I : x\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by Remark 2.8(ii), $(g \circ e) \in \hom_{\mathrm{S}}(\mathcal{A}, \mathcal{B})$ is injective, and so bijective, because $|A| = 3 \leqslant n$, for no $n \in 3 = |B|$.

– 2 forms a subalgebra of $\mathfrak{A}$,
in which case $\mathcal{E} \triangleq (\mathcal{A}{\restriction}2)$ is $\sim$-classical, while $a, b \in E^I$. Moreover, $a \in D^{\mathcal{D}} \not\ni b$, for $I \neq \varnothing$, while $\sim^{\mathfrak{D}}(a/b) = (b/a)$, in which case $\mathcal{F} \triangleq ((\mathcal{D}{\restriction}\{\sim\}){\restriction}\{a, b\})$ is $\sim$-classical (in particular, simple) with $0_{\mathcal{F}} = b$ and $1_{\mathcal{F}} = a$, whereas $(g{\restriction}F) \in \hom_{\mathrm{S}}(\mathcal{F}, \mathcal{B}{\restriction}\{\sim\})$, and so, by Remarks 2.8(ii) (implying the injectivity of $g{\restriction}F$) and 2.9(ii), $(\mathcal{B}{\restriction}\{\sim\}){\restriction}g[F]$ is $\sim$-classical, while $g(a) \in D^{\mathcal{B}} \not\ni g(b)$. Hence, $g(a) = 1_{\mathcal{B}}$ and $g(b) = 0_{\mathcal{B}}$. Then, $(\frac{1}{2})_{\mathcal{B}} \in B = g[D]$, in which case there is some $c \in D$ such that $g(c) = (\frac{1}{2})_{\mathcal{B}}$. Let $\mathcal{G}$ be the submatrix of $\mathcal{D}$ generated by $\{a, b, c\}$, in which case $f \triangleq (g{\restriction}G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{B})$, for $g[\{a, b, c\}] = B$. Let $J \triangleq \{i \in I \mid \pi_i(c) = \frac{1}{2}\}$, in which case $\pi_i(c) \in E$, for all $i \in (I \setminus J)$, and so, if J was empty, then c would be in E^I, in which case $\mathcal{G}$ would be a submatrix of $\mathcal{E}^I$, and so, by (2.23), C, being defined by $\mathcal{B}$, would be $\sim$-classical. Therefore, $J \neq \varnothing$. Take any $j \in J$. Let us prove, by contradiction, that $(\pi_j{\restriction}G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. For suppose $(\pi_j{\restriction}G) \notin \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. Then, as $(\pi_j{\restriction}G) \in \hom^{\mathrm{S}}(\mathcal{G}, \mathcal{A})$, there is some $d \in (G \setminus D^{\mathcal{G}})$ such that $\pi_j(d) \in D^{\mathcal{A}}$. Consider the following complementary subsubcases:

∗ $\mathcal{A}$ is not truth-singular.
Then, by Lemma 2.12 and Remark 2.8(ii), $\mathcal{A}$, being simple and finite, is a strict surjective homomorphic image of a subdirect product of a tuple constituted by submatrices of $\mathcal{B}$, in which case this is not truth-singular, and so is false-singular. Therefore, as $d \notin D^{\mathcal{G}}$, we have $f(d) \notin D^{\mathcal{B}}$, in which case $f(d) = 0_{\mathcal{B}}$, for $\mathcal{B}$ is false-singular, and so $\sim^{\mathfrak{B}} f(d) = 1_{\mathcal{B}} \in D^{\mathcal{B}}$. On the other hand, as $\mathcal{A}$ is not $\sim$-paraconsistent but is consistent, $\pi_j(\sim^{\mathfrak{G}} d) = \sim^{\mathfrak{A}} \pi_j(d) \notin D^{\mathcal{A}}$, in which case $\sim^{\mathfrak{G}} d \notin D^{\mathcal{G}}$, and so $\sim^{\mathfrak{B}} f(d) = f(\sim^{\mathfrak{G}} d) \notin D^{\mathcal{B}}$.

∗ $\mathcal{A}$ is truth-singular.
Then, $\pi_j(d) = 1_{\mathcal{A}} = \pi_i(d)$, for all $i \in J$, because $\pi_j(e) = \pi_i(e)$, for all $e \in \{a, b, c\}$, and so for all $e \in G \ni d$, in which case $d \in E^I \supseteq \{a, b\}$, and so the submatrix $\mathcal{H}$ of $\mathcal{G}$ generated by $\{a, b, d\}$ is a submatrix of $\mathcal{E}^I$. Moreover, $\pi_j(\sim^{\mathfrak{G}} d) = \sim^{\mathfrak{A}} \pi_j(d) = 0_{\mathcal{A}} \notin D^{\mathcal{A}}$, in which case $(\{d, \sim^{\mathfrak{G}} d\} \cap D^{\mathcal{G}}) = \varnothing$, and so $(\{f(d), \sim^{\mathfrak{B}} f(d)\} \cap D^{\mathcal{B}}) = \varnothing$. Hence, $f(d) = (\frac{1}{2})_{\mathcal{B}}$, in which case $f[\{a, b, d\}] = B$, and so $(f{\restriction}H) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{H}, \mathcal{B})$. In this way, by (2.23), C, being defined by $\mathcal{B}$, is $\sim$-classical.

Thus, anyway, we come to a contradiction. Therefore, $(\pi_j{\restriction}G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{A})$. Hence, since $f \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G}, \mathcal{B})$, by Remark 2.8(ii) and Lemma 2.11, $\mathcal{A}$ and $\mathcal{B}$, being both simple, are isomorphic.

[Then, Lemma 6.27 completes the argument.] □

In view of Corollary 6.29 [and Theorem 6.38], any [non-$\sim$-classical] three-valued Σ-logic with subclassical negation $\sim$ is defined by a [unique] canonical three-valued $\sim$-super-classical Σ-matrix [said to be *characteristic for/of* the logic], $\mathcal{A}$ being characteristic for C. On the other hand, the uniqueness is not, generally speaking, the case for $\sim$-classical (even both implicative — in particular, disjunctive — and conjunctive) ones, in view of Corollary 6.29 and Example 6.32.

Corollary 6.40. *Let $\Sigma' \supseteq \Sigma$ be a signature and C' a three-valued Σ'-expansion of C. Suppose $\mathcal{A}$ is both either false-singular or conjunctive or disjunctive and simple* *(i.e., C is not $\sim$-classical; cf. Corollary 6.36). Then, C' is defined by a unique Σ'-expansion of $\mathcal{A}$.*

Proof. In that case, $\sim$ is a subclassical negation for C'. Hence, by Corollary 6.29, C' is defined by a canonical three-valued $\sim$-super-classical Σ'-matrix $\mathcal{A}'$, in which case C is defined by the canonical three-valued $\sim$-super-classical Σ-matrix $\mathcal{A}'{\restriction}\Sigma$, and so, by Theorem 6.38, this is equal to $\mathcal{A}$. Finally, Lemma 6.27 and Theorem 6.38 complete the argument. □

And what is more, taking Lemma 6.5 into account, it is worth to explore connections between self-extensionality and existence of a classical extension. This makes the characterization to be obtained below especially acute. We start from exploring certain issues to be proved closely related to the primary one mentioned above.

A $(2[+1])$-*ary [$\frac{1}{2}$-relative] (classical) semi-conjunction for $\mathcal{A}$* is any $\varphi \in \mathrm{Fm}_\Sigma^{2[+1]}$ such that both $\varphi^{\mathfrak{A}}(0,1[,\frac{1}{2}]) = 0$ and $\varphi^{\mathfrak{A}}(1,0[,\frac{1}{2}]) \in \{0[,\frac{1}{2}]\}$. (Clearly, any binary semi-conjunction for $\mathcal{A}$ is a ternary $\frac{1}{2}$-relative one.)

Lemma 6.41. *Let $\mathcal{B}$ be a $\sim$-paraconsistent model of C. Suppose either $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction or $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$ or $\mathcal{B}$ is weakly $\sim$-negative or*

$$x_0 \vdash {\sim} x_0 \tag{6.13}$$

is not true in $\mathcal{B}$. Then, $\mathcal{A}$ is embeddable into a strict homomorphic image of a $\sim$-paraconsistent submatrix of $\mathcal{B}$.

Proof. Then, C is $\sim$-paraconsistent, and so is not $\sim$-classical, in which case, by Theorem 6.35(iv)⇒(i), $\mathcal{A}$ is simple. Moreover, [in case (6.13) is not true in $\mathcal{B}$] there are some $a, b[, c] \in B$ such that ${\sim}^{\mathfrak{B}} a[, c] \in D^{\mathcal{B}} \not\ni b[, {\sim}^{\mathfrak{B}} c]$. Therefore, by (2.23), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{a, b[, c]\}$ is a finitely-generated $\sim$-paraconsistent model of C [in which (6.13) is not true]. Hence, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{E}$ of it, some strict surjective homomorphic image $\mathcal{F}$ of $\mathcal{D}$ and some $h \in \hom_\mathrm{S}(\mathcal{E}, \mathcal{F})$, in which case, by (2.23), $\mathcal{E}$ is $\sim$-paraconsistent, and so consistent (in particular, $I \neq \varnothing$) [while (6.13) is not true in $\mathcal{E}$]. Given any $a' \in A$ and any $J \subseteq I$, set $(J : a') \triangleq (J \times \{a'\}) \in A^J$. Likewise, given any $\bar{a} \in A^2$ and any $J \subseteq I$, set $(a_0 \|_J a_1) \triangleq ((J : a_0) \cup ((I \setminus J) : a_1)) \in A^I$. Then, there are some $d \in (E \setminus D^{\mathcal{E}})$ and some $e[, f] \in D^{\mathcal{E}}$ such that ${\sim}^{\mathfrak{E}} e \in D^{\mathcal{E}} [\not\ni {\sim}^{\mathfrak{E}} f]$, in which case $e = (I : \frac{1}{2})$ and $J \triangleq \{i \in I \mid \pi_i(d) = 0\} \neq \varnothing [\neq K \triangleq \{i \in I \mid \pi_i(f) = 1\}$. Consider the following complementary cases:

- $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$,
  in which case ${\sim}^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. We are going to prove that there is some non-empty $L \subseteq I$ such that $(0 \|_L \frac{1}{2}) \in E$. For consider the following exhaustive subcases:
  - $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction φ.
    Let $g \triangleq \varphi^{\mathfrak{E}}(d, {\sim}^{\mathfrak{E}} d, e)$. Consider the following exhaustive subsubcases:
    * $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = 0$.
      Let $L \triangleq \{i \in I \mid \pi_i(d) \neq \frac{1}{2}\} \supseteq J$. Then, $E \ni g = (0 \|_L \frac{1}{2})$.
    * $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = \frac{1}{2}$.
      Let $L \triangleq J$. Then, $E \ni g = (0 \|_L \frac{1}{2})$.
  - $\mathcal{B}$ is weakly $\sim$-negative.
    Then, by Remark 2.9(ii), $\mathcal{E}$ is weakly $\sim$-negative, in which case ${\sim}^{\mathfrak{E}} d \in D^{\mathcal{E}}$, and so $d \in \{0, \frac{1}{2}\}^I$. Let $L \triangleq J$. Then, $E \ni d = (0 \|_L \frac{1}{2})$.
  - (6.13) is not true in $\mathcal{B}$.
    Let $L \triangleq K$. Then, $f \in D^{\mathcal{E}} \subseteq \{\frac{1}{2}, 1\}^I$, in which case $E \ni f = (1 \|_L \frac{1}{2})$, and so $E \ni {\sim}^{\mathfrak{E}} f = (0 \|_L \frac{1}{2})$.

In this way, $(0\|_L\frac{1}{2}) \in E \ni e = (\frac{1}{2}\|_L\frac{1}{2})$, in which case $E \ni \sim^{\mathfrak{E}}(0\|_L\frac{1}{2}) = (1\|_L\frac{1}{2})$, and so, as $L \neq \varnothing$, while $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $h' \triangleq \{\langle x, (x\|_K\frac{1}{2})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$.

- $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$,
  in which case there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}) \in 2$, and so $A = \{\frac{1}{2}, \varphi^{\mathfrak{A}}(\frac{1}{2}), \sim^{\mathfrak{A}}\varphi^{\mathfrak{A}}(\frac{1}{2})\}$. Hence, $\{I : x \mid x \in A\} = \{e, \varphi^{\mathfrak{E}}(e), \sim^{\mathfrak{E}}\varphi^{\mathfrak{E}}(e)\} \subseteq E$. Therefore, as $I \neq \varnothing$, $h' \triangleq \{\langle x, I : x\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$.

Thus, $(h \circ h') \in \mathrm{hom_S}(\mathcal{A}, \mathcal{F})$ is injective, in view of Remark 2.8(ii), as required. □

Theorem 6.42. *Suppose $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent) [and C is $\sim$-subclassical]. Then, the following are equivalent:*

(i) *C has no proper $\sim$-paraconsistent [$\sim$-subclassical] extension;*
(ii) *C has no proper $\sim$-paraconsistent non-$\sim$-subclassical extension;*
(iii) *either $\mathcal{A}$ has a ternary $\frac{1}{2}$-relative semi-conjunction or $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$ (in particular, $\sim^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$);*
(iv) *$L_3 \triangleq \{\langle\frac{1}{2}, \frac{1}{2}\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle\}$ does not form a subalgebra of $\mathfrak{A}^2$;*
(v) *$\mathcal{A}$ has no truth-singular $\sim$-paraconsistent subdirect square;*
(vi) *$\mathcal{A}^2$ has no truth-singular $\sim$-paraconsistent submatrix;*
(vii) *C has no truth-singular $\sim$-paraconsistent model;*
(viii) *$\mathcal{A}_{\frac{1}{2}}$ is not a $\sim$-paraconsistent model of C;*
(ix) *C has no truth-singular $\sim$-paraconsistent model over $\mathfrak{A}$.*

In particular, C has a $\sim$-paraconsistent proper extension iff it has a [non-]non-$\sim$-subclassical one, and if any three-valued expansion of C does so.

Proof. First, assume (iii) holds. Consider any $\sim$-paraconsistent extension C' of C, in which case $x_1 \notin T \triangleq C'(\{x_0, \sim x_0\}) \supseteq \{x_0, \sim x_0\}$, while, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so, by Lemma 6.41 and (2.23), $\mathcal{A}$ is a model of C', and so $C' = C$. Thus, both (i) and (ii) hold.

Next, assume L_3 forms a subalgebra of $\mathfrak{A}^2$. Then, by (2.23), $\mathcal{B} \triangleq (\mathcal{A}^2 \restriction L_3) \in \mathrm{Mod}(C)$ is a subdirect square of $\mathcal{A}$. Moreover, as $L_3 \ni \langle 0, 1\rangle \notin (L_3 \cap \Delta_A) = \{\langle\frac{1}{2}, \frac{1}{2}\rangle\} = D^{\mathcal{B}}$, for $\mathcal{A}$ is false-singular, $\mathcal{B}$ is both truth-singular and $\sim$-paraconsistent. Moreover, $(\pi_0 \restriction L_3) \in \mathrm{hom^S_S}(\mathcal{B}, \mathcal{A}_{\frac{1}{2}})$. Hence, by (2.23), $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$ is $\sim$-paraconsistent. Thus, (v/viii)$\Rightarrow$(iv) holds, while (v/viii/ix) is a particular case of (vi/ix/vii), whereas (vii)$\Rightarrow$(vi) is by (2.23).

Now, let $\mathcal{B} \in \mathrm{Mod}(C)$ be both $\sim$-paraconsistent and truth-singular, in which case (6.13) is true in $\mathcal{B}$, and so is its logical consequence

$$\{x_0, x_1, \sim x_1\} \vdash \sim x_0, \tag{6.14}$$

not being true in $\mathcal{A}$ under $[x_0/1, x_1/\frac{1}{2}]$ [but true in any $\sim$-classical model $\mathcal{C}'$ of C, for $\mathcal{C}'$ is $\sim$-negative]. Thus, the logic of $\{\mathcal{B}[, \mathcal{C}']\}$ is a proper $\sim$-paraconsistent [$\sim$-subclassical] extension of C, so (i)$\Rightarrow$(vii) holds. And what is more, (6.13), being true in $\mathcal{B}$, is not true in any $\sim$-[super-]classical Σ-matrix [in particular, in $\mathcal{A}$], in view of [(2.23) and] (3.11) with $n = 0$ and $m = 1$. Thus, the logic of $\mathcal{B}$ is a proper $\sim$-paraconsistent non-$\sim$-subclassical extension of C, so (ii)$\Rightarrow$(vii) holds.

Finally, assume $\mathcal{A}$ has no ternary $\frac{1}{2}$-relative semi-conjunction and $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$. In that case, $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by L_3. If $\langle 0, 0\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(0, 1, \frac{1}{2}) = 0 = \varphi^{\mathfrak{A}}(1, 0, \frac{1}{2})$, in which case it would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$. Likewise, if either $\langle\frac{1}{2}, 0\rangle$ or $\langle 0, \frac{1}{2}\rangle$ was in B, then there would be some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(0, 1, \frac{1}{2}) = 0$ and $\varphi^{\mathfrak{A}}(1, 0, \frac{1}{2}) = \frac{1}{2}$, in which case it would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$. Therefore, as $\sim^{\mathfrak{A}}1 = 0$ and $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, we conclude that $(\{\langle 0,\frac{1}{2}\rangle,\langle 1,\frac{1}{2}\rangle,\langle\frac{1}{2},1\rangle,\langle\frac{1}{2},0\rangle,\langle 0,0\rangle,\langle 1,1\rangle\}\cap B)=\varnothing$. Thus, $B=L_3$ forms a subalgebra of $\mathfrak{A}^2$. In this way, (iv)⇒(iii) holds.

After all, Corollary 6.40 completes the argument, for any expansion of $\mathcal{A}$ inherits ternary $\frac{1}{2}$-relative semi-conjunctions (if any). □

Theorem 6.42(i)⇔(iii[iv]) is especially useful for [effective dis]proving the maximal $\sim$-paraconsistency of C, as we show below [cf. Example 6.138]. And what is more, since, by Remark 2.9(i)**d)**, $\mathcal{A}$ has no proper $\sim$-paraconsistent submatrix, by Corollaries 2.14 and 6.29, we immediately have the following "axiomatic" version of Theorem 6.42:

Corollary 6.43. *Any [non-]non-$\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ has no $\sim$-paraconsistent [proper axiomatic] extension [and so is axiomatically maximally $\sim$-paraconsistent].*

Let $C_{\frac{1}{2}}$ be the logic of $\mathcal{A}_{\frac{1}{2}}$.

Lemma 6.44. *Let $\mathcal{B}\in\mathrm{Mod}(C)$. Suppose C is a non-purely-inferential $\sim$-paraconsistent sublogic of $C_{\frac{1}{2}}$. Then, $\mathcal{B}$ is consistent iff it is $\sim$-paraconsistent. In particular, $\mathcal{A}_{\frac{1}{2}}$ is $\sim$-paraconsistent.*

Proof. The "if" part is immediate. Conversely, assume $\mathcal{B}$ is consistent. Then, by the structurality of C, applying the Σ-substitution extending $[x_i/x_0]_{i\in\omega}$ to any theorem of C, we conclude that there is some $\phi\in(\mathrm{Fm}^1_\Sigma\cap C(\varnothing))$, and so, as $\mathcal{A}_{\frac{1}{2}}\in\mathrm{Mod}(C)$, $\phi^{\mathfrak{A}}(a)=\frac{1}{2}$, for all $a\in A$. Take any $b\in(B\setminus D^{\mathcal{B}})\neq\varnothing$, for $\mathcal{B}$ is consistent. Then, by (2.23), the submatrix $\mathcal{D}$ of $\mathcal{B}$ generated by $\{b\}$ is a finitely-generated consistent model of C. Hence, by Lemma 2.12, there are some set I and some submatrix $\mathcal{E}\in\mathbf{H}^{-1}(\mathbf{H}(\mathcal{D}))$ of $\mathcal{A}^I$. Take any $e\in E\neq\varnothing$. Then, $\phi^{\mathfrak{E}}(e)=(I\times\{\frac{1}{2}\})\in D^{\mathcal{E}}$, in which case $\sim^{\mathfrak{E}}\phi^{\mathfrak{E}}(e)\in D^{\mathcal{E}}$, for $\mathcal{A}$ is $\sim$-paraconsistent, and so $\mathcal{E}$, being consistent, for $\mathcal{D}$ is so, is $\sim$-paraconsistent. Thus, $\mathcal{B}$ is so, in view of (2.23), as required. □

Theorem 6.45. *Suppose C has a proper $\sim$-paraconsistent extension. Then, the following hold:*

(i) *$C_{\frac{1}{2}}$ is the proper ($\sim$-para)consistent extension of C relatively axiomatized by (6.13);*
(ii) *$C_{\frac{1}{2}}$ has no proper inferentially consistent (in particular, $\sim$-paraconsistent) extension;*
(iii) *the following are equivalent:*
 a) *C has a theorem;*
 b) *2 does not form a subalgebra of $\mathfrak{A}$;*
 c) *C is not $\sim$-subclassical;*
 d) *$C_{\frac{1}{2}}$ is the only proper ($\sim$-para)consistent extension of C;*
 e) *$C_{\frac{1}{2}}$ has no proper sublogic being a proper extension of C.*

In particular, any three-valued $\sim$-paraconsistent Σ-logic with subclassical negation $\sim$ is premaximally $\sim$-paraconsistent extension iff it is either maximally $\sim$-paraconsistent or not $\sim$-subclassical/purely-inferential (in particular, weakly disjunctive [in particular, implicative]).

Proof. Then, C is $\sim$-paraconsistent, and so is $\mathcal{A}$, in which case this is false-singular. Hence, by Theorem 6.42(iii/iv/viii)⇒(i), $\mathcal{A}_{\frac{1}{2}}\in\mathrm{Mod}(C)$ is $\sim$-paraconsistent, while $\mathcal{A}$ has no ternary $\frac{1}{2}$-relative semi-conjunction, whereas $\{\frac{1}{2}\}|L_3$ forms a subalgebra of $\mathfrak{A}|\mathfrak{A}^2$, respectively (in particular, $\sim^{\mathfrak{A}}\frac{1}{2}=\frac{1}{2}$).

(i) Then, (6.13), not being true in $\mathcal{A}$ under $[x_0/1]$, is true in $\mathcal{A}_{\frac{1}{2}}$. In this way, the logic of $\mathcal{A}_{\frac{1}{2}}$ is a proper ($\sim$-para)consistent extension of C satisfying (6.13).

Conversely, consider any Σ-rule $\Gamma \vdash \phi$ not satisfied in the extension C' of C relatively axiomatized by (6.13), in which case, as $\sim[\Gamma] \subseteq C'(\Gamma)$, the Σ-rule $(\Gamma \cup \sim[\Gamma]) \vdash \phi$ is not satisfied in C', and so in its sublogic C. Then, there is some $h \in \hom(\mathfrak{Fm}_\Sigma^\omega, \mathfrak{A})$ such that $h[\Gamma \cup \sim[\Gamma]] \subseteq D^{\mathcal{A}} = \{\frac{1}{2}, 1\} \not\ni h(\phi)$. In particular, $h(\phi) \neq \frac{1}{2}$. And what is more, for each $\psi \in \Gamma$, both $h(\psi) \in D^{\mathcal{A}}$ and $\sim^{\mathfrak{A}} h(\psi) = h(\sim\psi) \in D^{\mathcal{A}}$, in which case $h(\psi) = \frac{1}{2}$, for $\sim^{\mathfrak{A}} 1 = 0 \notin D^{\mathcal{A}}$, and so $h[\Gamma] \subseteq \{\frac{1}{2}\} = D^{\mathcal{A}_{\frac{1}{2}}} \not\ni h(\phi)$. Thus, $C' = C_{\frac{1}{2}}$.

(ii) Consider any inferentially consistent extension C' of $C_{\frac{1}{2}}$, in which case $x_1 \notin T \triangleq C'(x_0) \ni x_0$. Then, by the structurality of C', $\langle \mathfrak{Fm}_\Sigma^\omega, T\rangle$ is a model of C' (in particular, of $C_{\frac{1}{2}}$), and so is its finitely-generated consistent truth-non-empty submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}_\Sigma^2, T\rangle$, in view of (2.23). Hence, by Lemma 2.12, there are some set I and some submatrix $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of $\mathcal{A}_{\frac{1}{2}}^I$, in which case, by (2.23), $\mathcal{D}$ is a consistent truth-non-empty model of C', for $\mathcal{B}$ is so, and so $I \neq \varnothing$, while there are some $a \in D^{\mathcal{D}}$ and some $b \in (D \setminus D^{\mathcal{D}})$. Then, $D \ni a = (I \times \{\frac{1}{2}\}) \neq b$, in which case either $J \triangleq \{i \in I \mid \pi_i(b) = 1\}$ or $K \triangleq \{i \in I \mid \pi_i(b) = 0\}$ is non-empty. Given any $\bar{c} \in A^3$, set $(c_0 \| c_1 \| c_2) \triangleq ((J \times \{c_0\}) \cup (K \times \{c_1\}) \cup ((I \setminus (J \cup K)) \times \{c_2\})) \in A^I$. In this way, $D \ni a = (\frac{1}{2} \| \frac{1}{2} \| \frac{1}{2})$ and $D \ni b = (1 \| 0 \| \frac{1}{2})$, in which case $D \ni \sim^{\mathfrak{D}} b = (0 \| 1 \| \frac{1}{2})$. Consider the following exhaustive cases:

- $J \neq \varnothing \neq K$.
  Then, as $\{\frac{1}{2}\}|L_3$ forms a subalgebra of $\mathfrak{A}|\mathfrak{A}^2$, $\{\langle\langle x, y\rangle, (x \| y \| \frac{1}{2})\rangle \mid \langle x, y\rangle \in L_3\}$ is an embedding of $\mathcal{E} \triangleq (\mathcal{A}^2 \upharpoonright L_3)$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{E}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{A}_{\frac{1}{2}}$, for $(\pi_0 \upharpoonright L_3) \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{E}, \mathcal{A}_{\frac{1}{2}})$.
- $K = \varnothing$,
  in which case $J \neq \varnothing$, while $D \ni a = (\frac{1}{2} \| \frac{1}{2} \| \frac{1}{2})$, whereas $D \ni b = (0 \| \frac{1}{2} \| \frac{1}{2})$, and so $D \ni \sim^{\mathfrak{D}} b = (1 \| \frac{1}{2} \| \frac{1}{2})$. Then, as $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $\{\langle x, (x \| \frac{1}{2} \| \frac{1}{2})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}_{\frac{1}{2}}$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{A}_{\frac{1}{2}}$ is a model of C', for $\mathcal{D}$ is so.
- $J = \varnothing$,
  in which case $K \neq \varnothing$, while $D \ni a = (\frac{1}{2} \| \frac{1}{2} \| \frac{1}{2})$, whereas $D \ni b = (\frac{1}{2} \| 0 \| \frac{1}{2})$, and so $D \ni \sim^{\mathfrak{D}} b = (\frac{1}{2} \| 1 \| \frac{1}{2})$. Then, as $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, $\{\langle x, (\frac{1}{2} \| x \| \frac{1}{2})\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}_{\frac{1}{2}}$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{A}_{\frac{1}{2}}$ is a model of C', for $\mathcal{D}$ is so.

Thus, in any case, $\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C')$, and so $C' = C_{\frac{1}{2}}$.

(iii) First, assume **a)** holds. Consider any consistent extension C' of C, in which case $C'(\varnothing) \supseteq C(\varnothing) \neq \varnothing$, and so, if C' was inferentially inconsistent, then it, being structural, would be inconsistent, and the following complementary cases:

- (6.13) is satisfied in C',
  in which case, by (i), C' is an inferentially consistent extension of $C_{\frac{1}{2}}$, and so, by (ii), $C' = C_{\frac{1}{2}}$.
- (6.13) is not satisfied in C',
  in which case $\sim x_0 \notin T \triangleq C'(x_0) \ni x_0$. Then, by the structurality of C', $\mathcal{B} \triangleq \langle \mathfrak{Fm}_\Sigma^\omega, T\rangle$ is a model of C' (in particular, of C), in which (6.13) is not true under the diagonal Σ-substitution, in which case, by Lemma 6.44, $\mathcal{B}$, being consistent, is $\sim$-paraconsistent, for C is so, and so, by (2.23) and Lemma 6.41, $\mathcal{A}$ is a model of C', for $\mathcal{B}$ is so, in which case $C' = C$.

Thus, by (i), **d)** holds.

Next, **d)**⇒**e)** is by the consistency of $\mathcal{A}_{\frac{1}{2}}$, and so of $C_{\frac{1}{2}}$.

Now, let $\mathcal{B}$ be a $\sim$-classical (and so non-$\sim$-paraconsistent) model of C. Then, (6.14), being a logical consequence of $((2.16)[x_0/x_1, x_1/{\sim}x_0])/(6.13)$, is true in $\mathcal{B}/\mathcal{A}_{\frac{1}{2}}$, respectively/, in view of (i). However, it is not true in $\mathcal{A}$ under $[x_0/1, x_1/\frac{1}{2}]$. Moreover, by (3.11) with $n = 0$ and $m = 1$, (6.13) is not true in $\mathcal{B}$. In this way, by (i), the logic of $\{\mathcal{A}_{\frac{1}{2}}, \mathcal{B}\}$ is a proper extension/sublogic of $C_{/\frac{1}{2}}$. Thus, **e)**⇒**c)** holds.

Further, if 2 forms a subalgebra of $\mathfrak{A}$, then, by (2.23), $\mathcal{A}{\restriction}2$ is a $\sim$-classical model of C. Therefore, **c)**⇒**b)** holds.

Finally, assume **b)** holds. Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(1,0) = \frac{1}{2} = \varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})$, for $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$, in which case, if $\varphi^{\mathfrak{A}}(0,1)$ was equal to 0, then φ would be a ternary $\frac{1}{2}$-relative semi-conjunction for $\mathcal{A}$, and so $\varphi^{\mathfrak{A}}(0,1) \in D^{\mathcal{A}} \supseteq \{\varphi^{\mathfrak{A}}(1,0), \varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})\}$. In this way, $(\varphi[x_1/{\sim}x_0]) \in C(\varnothing)$, and so **a)** holds.

After all, Corollary 6.29 as well as Lemmas 6.30, 6.31 and the fact that (2.18) is a theorem of C, whenever $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent) and $\veebar$-disjunctive, complete the argument. □

In this way, Corollary 6.29 as well as Theorem[s] 6.42(i)⇔(iv) [and 6.45(iii)**b)**⇔**d)** provide an effective algebraic criterion of the [pre]maximal $\sim$-paraconsistency of three-valued $\sim$-paraconsistent Σ-logics with subclassical negation $\sim$.

Remark 6.46. Suppose either $\mathcal{A}$ is both false-singular and weakly $\bar{\wedge}$-conjunctive or both 2 forms a subalgebra of $\mathfrak{A}$ and $\mathcal{A}{\restriction}2$ is weakly $\bar{\wedge}$-conjunctive. Then, $(x_0 \mathbin{\bar{\wedge}} x_1)$ is a binary semi-conjunction for $\mathcal{A}$. □

By Corollary 6.29, Theorem 6.42(ii)⇒(i,ii) and Remark 6.46, we first have:

Corollary 6.47 (cf. the reference [Pyn 95b] of [17])**.** *Any weakly conjunctive three-valued Σ-logic with subclassical negation $\sim$ has no proper $\sim$-paraconsistent extension.*

The principal advance of this universal *maximal paraconsistency* result with regard to the reference [Pyn 95b] of [17] consists in extending the latter beyond subclassical logics towards those with merely subclassical negation, in which case, contrary to the latter, the former is equally applicable to arbitrary three-valued expansions (cf. Corollary 6.40 below in this connection) of logics under consideration, because expansions retain conjunction, subclassical negation and paraconsistency, but do not, generally speaking, inherit the property of being subclassical, and so the former, as opposed to the latter, covers *arbitrary three-valued expansions* of LP (being $\wedge$-conjunctive), HZ (being $\vee^{\sim}$-conjunctive) and P^1 (being conjunctive too; cf. [15]). In view of Example 6.67 below, the stipulation of the weak conjunctivity cannot be omitted in the formulation of Corollary 6.47.

Next, $\mathcal{A}$ is said to *satisfy Generation Condition (GC)*, provided either $\langle 0, 0\rangle$ or $\langle \frac{1}{2}, 0\rangle$ or $\langle 0, \frac{1}{2}\rangle$ belongs to the carrier of the subalgebra of $\mathfrak{A}^2$ generated by $\{\langle 1, \frac{1}{2}\rangle\}$. Put $M_2 \triangleq \{\langle 0, 1\rangle, \langle 1, 0\rangle\}$. Then, $\mathcal{A}$ is said to *satisfy Diagonal Generation Condition (DGC)*, provided Δ_A is not disjoint with the carrier of the subalgebra of $\mathfrak{A}^2$ generated by $M_2 \cup \{\langle 1, \frac{1}{2}\rangle\}$.

Lemma 6.48. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{D}$ a consistent truth-non-empty non-$\sim$-paraconsistent subdirect product of it. Suppose $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, while either $\mathcal{A}$ is either non-$\sim$-paraconsistent or weakly conjunctive, or $\mathcal{D}$ is $\sim$-negative or both $\mathcal{A}$ either has a binary semi-conjunction or satisfies GC, and either 2 forms a subalgebra of $\mathfrak{A}$ or $L_4 \triangleq (A^2 \setminus (2^2 \cup \{\frac{1}{2}\}^2))$ forms a subalgebra of $\mathfrak{A}^2$ or $\mathcal{A}$ satisfies DGC. Then, the following hold:*

(i) *if* 2 *forms a subalgebra of* $\mathfrak{A}$*, then* $\mathcal{A}{\restriction}2$ *is embeddable into* $\mathcal{D}$*;*
(ii) *if* 2 *does not form a subalgebra of* $\mathfrak{A}$*, then* $\mathcal{A}$ *is both* $\sim$*-paraconsistent (in particular, false-singular) and not weakly conjunctive, while* L_4 *forms a subalgebra of* $\mathfrak{A}^2$*, whereas* $\mathcal{A}^2{\restriction}L_4$ *is embeddable into* $\mathcal{D}$*.*

Proof. In that case, $I \neq \varnothing$, for $\mathcal{D}$ is consistent. Consider the following complementary cases:

(1) $(I \times \{i\}) \in D$, for some $i \in 2$,
in which case $D \ni \sim^{\mathfrak{D}}(I \times \{i\}) = (I \times \{1-i\})$, and so, if 2 did not form a subalgebra of $\mathfrak{A}$, then there would be some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(0,1) = \frac{1}{2}$, in which case D would contain $\varphi^{\mathfrak{D}}(I \times \{0\}, I \times \{1\}) = (I \times \{\frac{1}{2}\})$, and so, as $I \neq \varnothing$, $\{\langle a, I \times \{a\}\rangle \mid a \in A\}$ would be an embedding of $\mathcal{A}$ into $\mathcal{D}$ (in particular, by (2.23), $\mathcal{A}$ would be a model of the logic of $\mathcal{D}$). Therefore, 2 forms a subalgebra of $\mathfrak{A}$, in which case $\{\langle j, I \times \{j\}\rangle \mid j \in 2\}$ is an embedding of $\mathcal{A}{\restriction}2$ into $\mathcal{D}$, and so (i,ii) hold, in that case.

(2) $(I \times \{i\}) \in D$, for no $i \in 2$,
in which case, by Claim 6.39, $\mathcal{A}$ is both not weakly conjunctive and $\sim$-paraconsistent, and so false-singular. In particular,

$$e \triangleq (I \times \{\tfrac{1}{2}\}) \notin D, \tag{6.15}$$

for, otherwise, we would have $\{e, \sim^{\mathfrak{D}} e\} \subseteq D^{\mathcal{D}}$, contrary to the fact that $\mathcal{D}$ is not $\sim$-paraconsistent but is consistent. Take any $a \in D^{\mathcal{D}} \neq \varnothing$, for $\mathcal{D}$ is truth-non-empty, Then, $a \in \{\frac{1}{2}, 1\}^I$, in which case, by (2) with $i = 1$ and (6.15), $I \neq J \triangleq \{i \in I \mid \pi_i(a) = 1\} \neq \varnothing$, and so $b \triangleq \sim^{\mathfrak{D}} a \in (D \setminus D^{\mathcal{D}})$. Given any $\bar{a} \in A^2$, set $(a_0 \| a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. Then, $a = (1 \| \frac{1}{2})$.

Let us prove, by contradiction, that $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. For suppose $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$. Then, as $\mathcal{A}$ is $\sim$-paraconsistent, we have $\sim^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$, in which case we get $\sim^{\mathfrak{A}} \frac{1}{2} = 1$, and so both $b = (0 \| 1) \in D$ and $\sim^{\mathfrak{B}} b = (1 \| 0) \in D$ do not belong to $D^{\mathcal{D}}$, for $I \neq J \neq \varnothing$. Hence, $\mathcal{D}$ is not $\sim$-negative. Moreover, if $\mathcal{A}$ had a binary semi-conjunction φ, then D would contain $\varphi^{\mathfrak{A}}(b, \sim^{\mathfrak{B}} b) = (0 \| 0) = (I \times \{0\})$, contrary to (2) with $i = 0$. Likewise, if $\mathcal{A}$ satisfied GC, then there would be some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{A}}(\langle 1, \frac{1}{2}\rangle)$ would be in $\{\langle 0, \frac{1}{2}\rangle, \langle \frac{1}{2}, 0\rangle, \langle 0, 0\rangle\}$, in which case $\sim^{\mathfrak{A}} \psi^{\mathfrak{A}}(\langle 1, \frac{1}{2}\rangle)$ would be equal to $\langle 1, 1\rangle$, and so D would contain $\sim^{\mathfrak{D}} \psi^{\mathfrak{D}}(a) = (1 \| 1) = (I \times \{1\})$, contrary to (2) with $i = 1$. This contradicts to the fact that $\mathcal{A}$ is neither weakly conjunctive nor non-$\sim$-paraconsistent. Thus, $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$, in which case $b = (0 \| \frac{1}{2})$. Consider the following complementary subcases:

(i) 2 forms a subalgebra of $\mathfrak{A}$.
Let us prove, by contradiction, that so does $\{\frac{1}{2}\}$. For suppose $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, there is some $\psi \in \mathrm{Fm}^1_\omega$ such that $\psi^{\mathfrak{A}}(\frac{1}{2}) \in 2$, in which case $\psi^{\mathfrak{A}}[A] \subseteq 2$, for 2 forms a subalgebra of $\mathfrak{A}$, and so $\psi^{\mathfrak{A}} : A \to 2$ is not injective, for $|A| = 3 \nleq 2 = |2|$. Therefore, we have the following exhaustive subsubcases:
- $\psi^{\mathfrak{A}}(\frac{1}{2}) = \psi^{\mathfrak{A}}(0)$.
Then, $(I \times \{1\}) \in \{\psi^{\mathfrak{D}}(b), \sim^{\mathfrak{D}} \psi^{\mathfrak{D}}(b)\} \subseteq D$.
- $\psi^{\mathfrak{A}}(\frac{1}{2}) = \psi^{\mathfrak{A}}(1)$.
Then, $(I \times \{1\}) \in \{\psi^{\mathfrak{D}}(a), \sim^{\mathfrak{D}} \psi^{\mathfrak{D}}(a)\} \subseteq D$.
- $\psi^{\mathfrak{A}}(1) = \psi^{\mathfrak{A}}(0)$.
Then, $(I \times \{1\}) \in \{\psi^{\mathfrak{D}}(\psi^{\mathfrak{D}}(a)), \sim^{\mathfrak{D}} \psi^{\mathfrak{D}}(\psi^{\mathfrak{D}}(a))\} \subseteq D$.

Thus, anyway, $(I \times \{1\}) \in D$. This contradicts to (2) with $i = 1$. In this way, $\{\frac{1}{2}\}$ forms a subalgebra of $\mathfrak{A}$. Then, as $J \neq \varnothing$, while

$(1\|\frac{1}{2}) = a \in D \ni b = (0\|\frac{1}{2})$, $\{\langle i, (i\|\frac{1}{2})\rangle \mid i \in 2\}$ is an embedding of $\mathcal{A}{\restriction}2$ into $\mathcal{D}$.

(ii) 2 does not form a subalgebra of $\mathfrak{A}$.
Then, there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^\mathfrak{A}(0,1) = \frac{1}{2}$, in which case $\psi \triangleq \varphi[x_1/{\sim}x_0] \in \mathrm{Fm}^1_\Sigma$, while $\psi^\mathfrak{A}(0) = \varphi^\mathfrak{A}(0,1) = \frac{1}{2}$, and so, as $D \ni \psi^\mathfrak{D}(b)$, by (6.15), we have $\psi^\mathfrak{D}(\frac{1}{2}) \in 2$. Hence, we get $c \triangleq (\frac{1}{2}\|1) \in \{\psi^\mathfrak{D}(b), {\sim}^\mathfrak{D}\psi^\mathfrak{D}(b)\} \subseteq D$, in which case $D \ni d \triangleq {\sim}^\mathfrak{D}c = (\frac{1}{2}\|0)$, and so $\{(x\|y) \mid \langle x,y\rangle \in L_4\} = \{a,b,c,d\} \subseteq D$. Let us prove, by contradiction, that L_4 forms a subalgebra of $\mathfrak{A}^2$. For suppose L_4 does not form a subalgebra of $\mathfrak{A}^2$, in which case there is some $\phi \in \mathrm{Fm}^4_\Sigma$ such that $\phi^{\mathfrak{A}^2}(\langle 1,\frac{1}{2}\rangle, \langle 0,\frac{1}{2}\rangle, \langle\frac{1}{2},1\rangle, \langle\frac{1}{2},0\rangle) \in (A \setminus L_4) = (2^2 \cup \{\frac{1}{2}\}^2)$, and so $D \ni e \triangleq \phi^\mathfrak{D}(a,b,c,d) = (x\|y)$, where $\langle x,y\rangle \in (2^2 \cup \{\frac{1}{2}\}^2)$. Then, by (2) and (6.15), $\langle x,y\rangle \in (2^2 \setminus \Delta_2)$, in which case $0 \in \{x,y\}$, and so $e \in (D \setminus D^\mathcal{D}) \ni (y\|x) = {\sim}^\mathfrak{D}e$, for $I \neq J \neq \varnothing$. Hence, $\mathcal{D}$ is not ${\sim}$-negative. Therefore, $\mathcal{A}$ satisfies DGC, for it is is neither weakly conjunctive nor non-${\sim}$-paraconsistent, in which case there are some $\xi \in \mathrm{Fm}^3_\Sigma$ and some $z \in A$ such that $\xi^{\mathfrak{A}^2}(\langle 1,\frac{1}{2}\rangle, \langle 1,0\rangle, \langle 0,1\rangle) = \langle z,z\rangle$, and so $D \ni \xi^\mathfrak{D}(a, (1\|0), (0\|1)) = (z\|z)$, for $\{(1\|0), (0\|1)\} = \{e, {\sim}^\mathfrak{D}e\} \subseteq D \ni a$. This contradicts to (2) and (6.15). Therefore, L_4 forms a subalgebra of $\mathfrak{A}^2$. Hence, as $J \neq \varnothing \neq (I \setminus J)$, $\{\langle\langle x,y\rangle, (x\|y)\rangle \mid \langle x,y\rangle \in L_4\}$ is an embedding of $\mathcal{A}^2{\restriction}L_4$ into $\mathcal{D}$, as required. □

Corollary 6.49. *Let $\mathcal{B}$ be a ${\sim}$-classical model of C. Suppose C is not ${\sim}$-classical. Then, the following hold:*

(i) *if 2 forms a subalgebra of $\mathcal{A}$, then $\mathcal{A}{\restriction}2$ is isomorphic to $\mathcal{B}$;*
(ii) *if 2 does not form a subalgebra of $\mathcal{A}$, then both $\mathcal{B}$ is not disjunctive and C is both not weakly conjunctive and maximally ${\sim}$-paraconsistent, in which case $\mathcal{A}$ is ${\sim}$-paraconsistent, and so is false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\theta^{\mathcal{A}^2{\restriction}L_4} \in \mathrm{Con}(\mathfrak{A}^2{\restriction}L_4)$, $\langle \chi^{\mathcal{A}^2{\restriction}L_4}[\mathfrak{A}^2{\restriction}L_4], \{1\}\rangle$ being isomorphic to $\mathcal{B}$.*

Proof. Then, $\mathcal{B}$ is finite and simple. Therefore, by Lemma 2.12 and Remark 2.8(ii), there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it and some $h \in \hom^\mathrm{S}_\mathrm{S}(\mathcal{D},\mathcal{B})$, in which case, by Remark 2.9(ii), $\mathcal{D}$ is ${\sim}$-negative, for $\mathcal{B}$ is so, and so both consistent and truth-non-empty, while, by (2.23), the logic C' of $\mathcal{D}$ is the ${\sim}$-classical (in particular, non-${\sim}$-paraconsistent) one of $\mathcal{B}$, and so, by Corollary 3.33, $\mathcal{A}$, being both consistent and truth-non-empty, in which case C is inferentially-consistent, is not a model of C'. Consider the following complementary cases:

(i) 2 forms a subalgebra of $\mathfrak{A}$.
Then, by Lemma 6.48(i), there is some embedding e of $\mathcal{A}{\restriction}2$ into $\mathcal{D}$, in which case, by Remark 2.8(ii), $h \circ e$ is that into $\mathcal{B}$, and so is an isomorphism from $\mathcal{A}{\restriction}2$ onto $\mathcal{B}$, for this has no proper submatrix.

(ii) 2 does not form a subalgebra of $\mathfrak{A}$.
Then, by Theorem 6.45(iii)**b)⇒c)** and Lemma 6.48(ii), C is both not weakly conjunctive and maximally ${\sim}$-paraconsistent, in which case $\mathcal{A}$ is ${\sim}$-paraconsistent, and so is false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas there is some embedding e of of $\mathcal{F} \triangleq (\mathcal{A}^2{\restriction}L_4)$ into $\mathcal{D}$, in which case $g \triangleq (h \circ e) \in \hom^\mathrm{S}_\mathrm{S}(\mathcal{F},\mathcal{B})$, for $\mathcal{B}$, being ${\sim}$-classical, has no proper submatrix, and so, by Remark 2.8, $(\ker\chi^\mathcal{F}) = \theta^\mathcal{F} = g^{-1}[\theta^\mathcal{B}] = g^{-1}[\Delta_B] = (\ker g) \in \mathrm{Con}(\mathfrak{F})$, in which case $\chi^\mathcal{F}$ is a strict surjective homomorphism from $\mathcal{F}$ onto $\mathcal{G} \triangleq \langle \chi^\mathcal{F}[\mathfrak{F}], \{1\}\rangle$, and so, by the Homomorphism Theorem, $\chi^\mathcal{F} \circ g^{-1}$ is an

isomorphism from $\mathcal{B}$ onto $\mathcal{G}$. Finally, let us prove, by contradiction, that $\mathcal{B}$ is not disjunctive. For suppose $\mathcal{B}$ is $\veebar$-disjunctive, and so is $\mathcal{F}$, in view of Remark 2.9(ii). Then, as $\langle\frac{1}{2},1\rangle \in D^{\mathcal{F}}$, for $\mathcal{A}$ is false-singular, we have $\{\langle 0,\frac{1}{2}\rangle \veebar^{\mathfrak{F}} \langle\frac{1}{2},1\rangle, \langle\frac{1}{2},1\rangle \veebar^{\mathfrak{F}} \langle 0,\frac{1}{2}\rangle\} \subseteq D^{\mathcal{F}}$, in which case we get $\{0 \veebar^{\mathfrak{A}} \frac{1}{2}, \frac{1}{2} \veebar^{\mathfrak{A}} 0\} \subseteq D^{\mathcal{A}}$, and so we eventually get $(\langle 0,\frac{1}{2}\rangle \veebar^{\mathfrak{F}} \langle\frac{1}{2},0\rangle) \in D^{\mathcal{F}}$. This contradicts to the fact that $(\{\langle 0,\frac{1}{2}\rangle, \langle\frac{1}{2},0\rangle\} \cap D^{\mathcal{F}}) = \varnothing$. Thus, $\mathcal{B}$ is not disjunctive. □

Combining [Lemmas 3.32, 6.30 and] Corollary 6.49 with (2.23) [and Remark 2.9(ii)], we immediately get:

Theorem 6.50. *C has a [$\veebar$-disjunctive] $\sim$-classical extension iff either of the following [but (iii)] holds:*

(i) *C is $\sim$-classical [and $\veebar$-disjunctive];*

(ii) *2 forms a subalgebra of $\mathfrak{A}$ [with $\veebar$-disjunctive $\mathcal{A}{\restriction}2$], in which case $\mathcal{A}{\restriction}2$ is a canonical $\sim$-classical model of C isomorphic to any $\sim$-classical model of C, and so is a unique canonical one and defines a unique $\sim$-classical extension of C;*

(iii) *C is both not weakly conjunctive and maximally $\sim$-paraconsistent, in which case $\mathcal{A}$ is $\sim$-paraconsistent, and so is false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\theta^{\mathcal{A}^2{\restriction}L_4} \in \mathrm{Con}(\mathfrak{A}^2{\restriction}L_4)$, in which case $\langle\chi^{\mathfrak{A}^2{\restriction}L_4}[\mathfrak{A}^2{\restriction}L_4], \{1\}\rangle$ is a canonical $\sim$-classical model of C isomorphic to any $\sim$-classical model of C, and so is a unique canonical one and defines a unique $\sim$-classical extension of C.*

In view of Lemma 3.32 and Theorem 6.50, C, being $\sim$-subclassical, has a unique $\sim$-classical extension/"canonical model" to be denoted by $C^{\mathrm{PC}}/\mathcal{A}_{\mathrm{PC}}$, respectively, and referred to as *characteristic of|for C*, in which case $C^{\mathrm{PC}} = [\neq]C$, whenever C is [not] $\sim$-classical. It is remarkable that the $\veebar$-disjunctivity of C is not required in the []-optional version of Theorem 6.50, making this the right characterization of C's being *genuinely* $\sim$-subclassical in the sense of having a functionally complete $\sim$-classical extension. And what is more, by Lemma 6.30 and Theorem 6.50, we have:

Corollary 6.51. *[Suppose $\mathcal{A}$ is either truth-singular or weakly conjunctive or disjunctive (in particular, implicative).] Then, C is $\sim$-subclassical if[f] either of the following holds:*

(i) *C is $\sim$-classical;*

(ii) *2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\restriction}2$ is a canonical $\sim$-classical model of C isomorphic to any $\sim$-classical model of C, and so is a unique canonical one and defines a unique $\sim$-classical extension of C.*

The ([]-optional) stipulation in the formulation of Corollary 6.51 (resp., Theorem 6.50) cannot be omitted {or, even, "weakened"}, because of existence of three-valued {even, weakly disjunctive} non-$\sim$-classical ⟨even, $\sim$-paraconsistent⟩ $\sim$-subclassical Σ-logics, the underlying algebras of the characteristic matrices of which do not have subalgebras with carrier 2, as it ensues from:

Example 6.52. Let $i \in 2$, $\Sigma \triangleq \{\amalg, \sim\}$ with binary $\amalg$, $\mathcal{B}$ the canonical $\sim$-classical Σ-matrix with $(j \amalg^{\mathfrak{B}} k) \triangleq i$, for all $j,k \in 2$, and $\mathcal{A}$ false-singular with $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ and

$$(a \amalg^{\mathfrak{A}} b) \triangleq \begin{cases} i & \text{if } a = \frac{1}{2}, \\ \frac{1}{2} & \text{otherwise}, \end{cases}$$

for all $a,b \in A$, in which case $\mathcal{A}$ is both $\sim$-paraconsistent and, providing $i = 1$, weakly $\amalg$-disjunctive, and so is C. Then, we have:

$$(\langle\tfrac{1}{2},a\rangle \amalg^{\mathfrak{A}^2} \langle b,\tfrac{1}{2}\rangle) \quad = \quad \langle i,\tfrac{1}{2}\rangle,$$

$$\begin{array}{rcl} (\langle b, \frac{1}{2}\rangle \amalg^{\mathfrak{A}^2} \langle \frac{1}{2}, a\rangle) & = & \langle \frac{1}{2}, i\rangle, \\ (\langle \frac{1}{2}, a\rangle \amalg^{\mathfrak{A}^2} \langle \frac{1}{2}, b\rangle) & = & \langle i, \frac{1}{2}\rangle, \\ (\langle a, \frac{1}{2}\rangle \amalg^{\mathfrak{A}^2} \langle b, \frac{1}{2}\rangle) & = & \langle \frac{1}{2}, i\rangle, \end{array}$$

for all $a, b \in 2$. Hence, L_4 forms a subalgebra of $\mathfrak{A}^2$, while $\chi^{\mathcal{A}^2 \upharpoonright L_4} \in \hom_{\mathrm{S}}^{\mathrm{S}}(\mathcal{A}^2 \upharpoonright L_4, \mathcal{B})$, in which case, by (2.23), $\mathcal{B} \in \mathrm{Mod}(C)$, and so C is $\sim$-subclassical. However, $(0 \amalg^{\mathfrak{A}} 1) = \frac{1}{2}$, in which case 2 does not form a subalgebra of $\mathfrak{A}$, and so, by Corollary 6.51, C is neither disjunctive nor weakly conjunctive. □

Corollary 6.53. *Suppose $\mathcal{A}$ is $\sqsupset$-implicative (viz., C is so; cf. Lemma 6.31). Then, C has a proper consistent axiomatic extension iff it is non-$\sim$-classical (in particular, $\sim$-paraconsistent) and $\sim$-subclassical, in which case C^{PC} is a unique proper consistent axiomatic extension of C and is relatively axiomatized by $\bar{\phi} \sqsupset \psi$, where $\bar{\phi} \in (\mathrm{Fm}^1_\Sigma)^*$ and $\psi \in (C^{\mathrm{PC}}(\mathrm{img}\,\bar{\phi}) \setminus C(\mathrm{img}\,\bar{\phi}))$ (in particular, by (2.17)).*

Proof. According to Corollary 2.14, any [proper] {consistent} axiomatic extension of C is defined by some {non-empty} $\mathsf{S} \subseteq \mathbf{S}_*(\mathcal{A})$ [not containing $\mathcal{A}$, in which case $\mathsf{S} \subseteq \{=\}\{\mathcal{A} \upharpoonright 2\}$, if 2 forms a subalgebra of $\mathfrak{A}$, and $\mathsf{S} = \varnothing$, otherwise {and so (2.12), Corollaries 2.14, 3.33, 6.51 and Remark 2.9(ii)(,(i)**d**) complete the argument}. □

Lemma 6.54. *Let S be a set of Σ-matrices and C' the logic of S. Then, the following are equivalent:*

(i) *C' has a theorem;*
(ii) *for any set I, any $e \in \mathsf{S}^I$, any function f with domain I, and any $S \subseteq \prod_{i \in I} \lceil e(i) \rceil^{f(i)}$, the submatrix of $\prod_{i \in I} e(i)^{f(i)}$ generated by S is truth-non-empty;*
(iii) *for any set I, any $e \in \mathsf{S}^I$, any function f with domain I, and any $\vec{\bar{g}} \in \prod_{i \in I} \lceil e(i) \rceil^{f(i)}$, the submatrix of $\prod_{i \in I} e(i)^{f(i)}$ generated by $\{\vec{\bar{g}}\}$ is truth-non-empty;*
(iv) *for any enumeration e of S and any $|\mathsf{S}|$-tuple $\vec{\bar{g}}$ such that, for every $i \in |\mathsf{S}|$, $\bar{g}^i$ is an enumeration of $\lceil e(i) \rceil$, the submatrix of $\prod_{i \in I} e(i)^{|\lceil e(i) \rceil|}$ generated by $\{\vec{\bar{g}}\}$ is truth-non-empty.*

Proof. First, (i)⇒(ii) is by (2.23) and Corollary 3.21(ii)⇒(i). Next, (iii/iv) is a particular case of (ii/iii), respectively. Finally, assume (iv) holds. Take any enumeration e of S and, for each $i \in |\mathsf{S}|$, any enumeration $\bar{g}^i$ of $\lceil e(i) \rceil$. Let $\mathcal{D}$ be the submatrix of $\prod_{i \in I} e(i)^{|\lceil e(i) \rceil|}$ generated by $\{\vec{\bar{g}}\}$. Then, $D^{\mathcal{D}} \neq \varnothing$, in which case there is some $\varphi \in \mathrm{Fm}^1_\Sigma$ such that $\varphi^{\mathfrak{D}}(\vec{\bar{g}}) \in D^{\mathcal{D}}$, and so for each $i \in |\mathsf{S}|$ and every $j \in |\lceil e(i) \rceil|$, $\varphi^{\lceil e(i) \rceil}(g^i_j) = \pi_j(\pi_i(\varphi^{\mathfrak{D}}(\vec{\bar{g}}))) \in D^{\lceil e(i) \rceil}$. In this way, $\varphi \in C'(\varnothing)$. Thus, (i) holds. □

In case both S and all members of it are finite, Lemma 6.54(i)⇔(iii) provides an effective algebraic criterion of C''s having a theorem.

A *semi-conjunction for/of* a canonical $\sim$-classical Σ-matrix $\mathcal{B}$ is any $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(i, 1-i) = 0$, for all $i \in 2$.

Corollary 6.55. *Let $\mathcal{B}$ be a canonical $\sim$-classical Σ-matrix and C' the logic of $\mathcal{B}$. Then, the following are equivalent:*

(i) *C' has a theorem;*
(ii) *M_2 does not form a subalgebra of $\mathfrak{B}^2$;*
(iii) *$\mathcal{B}$ has a semi-conjunction.*

Proof. First, given any semi-conjunction φ of $\mathcal{B}$, $\sim\varphi[x_1/\sim x_0]$ is a theorem of C'. Hence, (iii)⇒(i) holds.

Next, assume (ii) holds. Then, there are some $\phi \in \mathrm{Fm}^2_\Sigma$ and some $j \in 2$ such that $\phi^{\mathfrak{B}}(i, 1-i) = j$, for all $i \in 2$, in which case $\sim^j\phi$ is a semi-conjunction of $\mathcal{B}$, and so (iii) holds.

Finally, assume (i) holds. Then, by Lemma 6.54(i)⇒(ii), the submatrix $\mathcal{D}$ of $\mathcal{B}^2$ generated by M_2 is truth-non-empty, in which case the unique distinguished value $\langle 1, 1\rangle \notin M_2$ of $\mathcal{B}^2$ belongs to D, and so $D \neq M_2$. Thus, (ii) holds. $\square$

Lemma 6.56. *Suppose C is $\sim$-subclassical. Then, the following are equivalent:*

(i) *C^{PC} has a theorem;*
(ii) *$\mathcal{A}$ has a binary semi-conjunction;*
(iii) *$M^{0/1}_{2[+2(+4)]}$ does not form a subalgebra of $(\mathfrak{A}^{([2])}(\upharpoonright L_{2[+2]}))^2$, whenever $L_2 \triangleq 2$ does [not] form a subalgebra of $\mathfrak{A}$, while $\theta^{\mathcal{A}} \in (\notin) \operatorname{Con}(\mathcal{A})$, whereas $\mathcal{A}$ is false-/truth-singular, where, for all $i \in 2$, $M^i_2 \triangleq M_2$, $M^i_4 \triangleq (M_2 \cup \{\langle i, \frac{1}{2}\rangle, \langle \frac{1}{2}, i\rangle\})$ and $M^{\{i\}}_8 \triangleq \{\langle\{\langle j, \frac{1}{2}\rangle, \langle 1-j, l\rangle\}, \{\langle k, \frac{1}{2}\rangle, \langle 1-k, 1-l\rangle\}\rangle \mid j, k, l \in 2\}$.*

Proof. Let $\mathcal{B} \triangleq \mathcal{A}_{\mathrm{PC}}$. Consider the following complementary cases:

- C is $\sim$-classical,
  in which case it is defined by $\mathcal{B}$, and so, by Lemma 3.32, there are some submatrix $\mathcal{D}$ of $\mathcal{A}$ and some $g \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$. Then, $\mathcal{D}$ is both consistent and truth-non-empty, for $\mathcal{B}$ is so, and so is not one-valued. Hence, $2 \subseteq D$. Assume $\mathcal{A}$ is false-/truth-singular. Then, both $\mathcal{B}$ and $\mathcal{D}$ are so with the unique non-distinguihed/distinguished value 0/1, in which case $g(0/1) = (0/1)$, and so $(1/0) = \sim^{\mathfrak{B}}(0/1) = \sim^{\mathfrak{B}} g(0/1) = g(\sim^{\mathfrak{D}}(0/1)) = g(\sim^{\mathfrak{A}}(0/1)) = g(1/0)$. Thus, $g(i) = i$, for all $i \in 2$. Consider the following complementary subcases:
  - 2 forms a subalgebra of $\mathfrak{A}$,
    and so of $\mathfrak{D}$, for $2 \subseteq D$, in which case $g\upharpoonright 2$ is a diagonal strict homomorphism from $(\mathcal{D}\upharpoonright 2) = (\mathcal{A}\upharpoonright 2)$ onto $\mathcal{B}$. Hence, $\mathcal{B} = (\mathcal{A}\upharpoonright 2)$. In particular, semi-conjunctions of $\mathcal{B}$ are exactly binary semi-conjunctions for $\mathcal{A}$. Moreover, $M_2 \subseteq 2^2$ forms a subalgebra of $\mathfrak{B}^2$, being a subalgebra of $\mathfrak{A}^2$, iff it forms a subalgebra of $\mathfrak{A}^2$.
  - 2 does not form a subalgebra of $\mathfrak{A}$.
    Then, $\mathcal{D} = \mathcal{A}$, for $2 \subseteq D$. Therefore, as $\mathcal{B}$ is truth-/false-singular, $g(\frac{1}{2}) = (1/0) = g(1/0)$, in which case g is not injective, and so, by Remark 2.8(ii) and Theorem 6.35(iii)⇒(v), $\theta^{\mathcal{A}} \in \operatorname{Con}(\mathfrak{A})$. Moreover, $f \triangleq ((g \circ (\pi_0 \upharpoonright A^2)) \times (g \circ (\pi_1 \upharpoonright A^2))) \in \hom(\mathfrak{A}^2, \mathfrak{B}^2)$ is surjective. Hence, M_2 forms a subalgebra of $\mathfrak{B}^2$ iff $M_4^{0/1} = f^{-1}[M_2]$ forms a subalgebra of $\mathfrak{A}^2$. Next, given any binary semi-conjunction φ for $\mathcal{A}$ and any $i \in 2$, we have $\varphi^{\mathfrak{A}}(i, 1-i) = 0$, in which case we get $\varphi^{\mathfrak{B}}(i, 1-i) = \varphi^{\mathfrak{B}}(g(i), g(1-i)) = g(\varphi^{\mathfrak{A}}(i, 1-i)) = g(0) = 0$, and so φ is a semi-conjunction of $\mathcal{B}$. Conversely, consider any semi-conjunction φ of $\mathcal{B}$, in which case, for all $i \in 2$, $g((\sim^{\mathfrak{A}})^{0/1}\varphi^{\mathfrak{A}}(i, 1-i)) = (\sim^{\mathfrak{B}})^{0/1}\varphi^{\mathfrak{B}}(g(i), g(1-i))) = (\sim^{\mathfrak{B}})^{0/1}\varphi^{\mathfrak{B}}(i, 1-i) = (\sim^{\mathfrak{B}})^{0/1}0 = (0/1) \notin / \in D^{\mathcal{B}}$, and so $(\sim^{\mathfrak{A}})^{0/1}\varphi^{\mathfrak{A}}(i, 1-i) \notin / \in D^{\mathcal{A}}$, in which case $(\sim^{\mathfrak{A}})^{0/1}\varphi^{\mathfrak{A}}(i, 1-i) = (0/1)$, and so $(\sim^{\mathfrak{A}})^{0/2}\varphi^{\mathfrak{A}}(i, 1-i) = (\sim^{\mathfrak{A}})^{0/1}(\sim^{\mathfrak{A}})^{0/1}\varphi^{\mathfrak{A}}(i, 1-i) = (\sim^{\mathfrak{A}})^{0/1}(0/1) = 0$. In this way, $\sim^{0/2}\varphi$ is a binary semi-conjunction for $\mathcal{A}$.
- C is not $\sim$-classical,
  in which case, by Theorem 6.35(v)⇒(i), $\theta^{\mathcal{A}} \notin \operatorname{Con}(\mathcal{A})$. Consider the following complementary subcases:

– 2 forms a subalgebra of $\mathfrak{A}$,
in which case $\mathcal{B} = (\mathcal{A}\restriction 2)$, in view of (2.23) and Theorem 6.50, and so binary semi-conjunctions for $\mathcal{A}$ are exactly semi-conjunctions of $\mathcal{B}$.

– 2 does not form a subalgebra of $\mathfrak{A}$.
Then, by Theorem 6.50, $\mathcal{A}$ is false-singular, while L_4 forms a subalgebra of $\mathfrak{A}^2$, whereas $\theta^{\mathcal{A}^2\restriction L_4} \in \mathrm{Con}(\mathfrak{A}^2\restriction L_4)$, in which case $\mathcal{B} = \langle h[\mathfrak{A}^2\restriction L_4], \{1\}\rangle$, where $h \triangleq \chi^{\mathfrak{A}^2\restriction L_4}$ is a strict surjective homomorphism from $\mathcal{D} \triangleq (\mathcal{A}^2\restriction L_4)$ onto $\mathcal{B}$, and so $g \triangleq ((h \circ (\pi_0\restriction D^2) \times (h \circ (\pi_1\restriction D^2)) \in \mathrm{hom}(\mathfrak{D}^2, \mathfrak{B}^2)$ is surjective. In particular, M_2 forms a subalgebra of $\mathfrak{B}^2$ iff $M_8 = g^{-1}[M_2]$ forms a subalgebra of $\mathfrak{D}^2$. Moreover, as $\frac{1}{2} \in D^{\mathcal{A}}$, for $\mathcal{A}$ is truth-singular, $a \triangleq \langle 1, \frac{1}{2}\rangle \in D^{\mathcal{D}} \not\ni b \triangleq \langle 0, \frac{1}{2}\rangle \in D$, in which case we have $h(a|b) \in | \notin D^{\mathcal{B}}$, and so $h(a|b) = (1|0)$. Consider any binary semi-conjunction φ for $\mathcal{A}$. Then, $D \ni \varphi^{\mathfrak{D}}(a|b, b|a) = \varphi^{\mathfrak{A}^2}(a|b, b|a)$, in which case, as $(\pi_0\restriction A^2) \in \mathrm{hom}(\mathfrak{A}^2, \mathfrak{A})$, we have $\pi_0(\varphi^{\mathfrak{D}}(a|b, b|a)) = \varphi^{\mathfrak{A}}(\pi_0(a|b), \pi_0(b|a)) = \varphi^{\mathfrak{A}}(1|0, 0|1) = 0$, and so $\varphi^{\mathfrak{D}}(a|b, b|a) \notin D^{\mathcal{D}}$. Hence, $\varphi^{\mathfrak{B}}(1|0, 0|1) = \varphi^{\mathfrak{B}}(h(a|b), h(b|a)) = h(\varphi^{\mathfrak{D}}(a|b, b|a)) \notin D^{\mathcal{B}}$, in which case $\varphi^{\mathfrak{B}}(1|0, 0|1) = 0$, and so φ is a semi-conjunction of $\mathcal{B}$. Conversely, consider any semi-conjunction φ of $\mathcal{B}$. Then, $h(\varphi^{\mathfrak{D}}(a|b, b|a)) = \varphi^{\mathfrak{B}}(h(a|b), (b|a)) = \varphi^{\mathfrak{B}}(1|0, 0|1) = 0 \notin D^{\mathcal{B}}$, in which case $\langle \varphi^{\mathfrak{A}}(1|0, 0|1), \varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2})\rangle = \varphi^{\mathfrak{D}}(a|b, b|a) \notin D^{\mathcal{D}}$. Consider the following complementary subsubcases:

  * $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.
  Then, as $\frac{1}{2} \in D^{\mathcal{A}}$, for $\mathcal{A}$ is false-singular, $\varphi^{\mathfrak{A}}(1|0, 0|1) = 0$, and so φ is a binary semi-conjunction for $\mathcal{A}$.
  * $\varphi^{\mathfrak{D}}(\frac{1}{2}, \frac{1}{2}) \neq \frac{1}{2}$.
  Then, as 2^2 is disjoint with $L_4 = D \ni \varphi^{\mathfrak{D}}(a|b, b|a)$, $\varphi^{\mathfrak{A}}(1|0, 0|1) = \frac{1}{2}$, in which case, as $\frac{1}{2} \in D^{\mathcal{A}}$, for $\mathcal{A}$ is false-singular, $\varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) = 0$, and so $\varphi[x_i/\varphi]_{i\in 2}$ is a binary semi-conjunction for $\mathcal{A}$.

In this way, Corollary 6.55 completes the argument. □

Corollary 6.57. *Suppose C is $\sim$-subclassical and and weakly $\veebar$-disjunctive. Then, $\mathcal{A}$ has a binary semi-conjunction.*

Proof. In that case, $C^{\mathrm{PC}} \supseteq C$ is weakly $\veebar$-disjunctive, and so, by Remark 2.9(i)**d)**, satisfies (2.18). In this way, Lemma 6.56(i)⇒(ii) completes the argument. □

By Corollaries 6.29, 6.57, Lemmas 6.30, 6.31 and Theorem{s} 6.42(iii)⇒(i) [including the last assertion] {and 6.45}, we get the following "disjunctive" analogue of Corollary 6.47, being essentially beyond the scopes of the reference [Pyn 95b] of [17], and so becoming a one more substantial advance of the present study with regard to that one:

Corollary 6.58. *Any [three-valued expansion of any] disjunctive (in particular, implicative) {non-}$\sim$-subclassical three-valued Σ-logic {with subclassical negation $\sim$} has no {more than one} proper $\sim$-paraconsistent extension. In particular, any disjunctive (in particular, implicative) $\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ is premaximally $\sim$-paraconsistent.*

This (more precisely, the {}-non-optional part) is *immediately* applicable to arbitrary (not necessarily $\sim$-subclassical) three-valued expansions of the implicative $\sim$-subclassical P^1 and HZ. On the other hand, as opposed to Corollary 6.47, the condition of being $\sim$-subclassical in the formulation of the {}-non-optional part of Corollary 6.58 is essential, as it follows from Example 6.138 below.

Theorem 6.59. *Suppose $\mathcal{A}$ is [not] false-singular, while C is $\sim$-subclassical. Then, the following are equivalent:*

(i) *C has a theorem;*
(ii) *C^{PC} has a theorem [and $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$];*
(iii) *$\mathcal{A}$ has a binary semi-conjunction [and $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$];*
(iv) *[$\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$, and] providing L_2 does (not) form a subalgebra of $\mathfrak{A}$, while $\theta^{\mathcal{A}} \in \{\not\in\} \operatorname{Con}(\mathcal{A})$, whereas $\mathcal{A}$ is false-/truth-singular, $M^{0/1}_{2(+2\{+4\})}$ does not form a subalgebra of $(\mathfrak{A}^{\{(2)\}}\{\restriction L_{2(+2)}\})^2$;*
(v) *Any consistent extension of C is a sublogic of C^{PC}.*

Proof. First, the equivalence of (ii–iv) is by Lemma 6.56. Next, (i)⇒(ii) is by the fact that $C(\varnothing) \subseteq C^{\mathrm{PC}}(\varnothing)$ [as well as both (2.23) and Corollary 3.21(ii)⇒(i), for $\frac{1}{2} \notin D^{\mathcal{A}}$]. Conversely, assume (ii,iii) hold. Then, in case C is $\sim$-classical, and so $C = C^{\mathrm{PC}}$, (i) is a particular case of (ii). Otherwise, (i) is by (iii) and the following claim:

Claim 6.60. *Let φ be a binary semi-conjunction for $\mathcal{A}$. Suppose either $\mathcal{A}$ is false-singular or both C is $\sim$-subclassical but not $\sim$-classical, and $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$. Then, C has a theorem.*

Proof. Let $\mathcal{D}$ the submatrix of $\mathcal{A}^3$ generated by the enumeration $a \triangleq (10\frac{1}{2})$ of A. Consider the following complementary cases:

- $\mathcal{A}$ is false-singular.
  Consider the following exhaustive subcases:
  - $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.
    Then, $D \ni b \triangleq \sim^{\mathfrak{D}} a = (01\frac{1}{2})$. Let $x \triangleq \varphi^{\mathfrak{A}}(\frac{1}{2}, \frac{1}{2}) \in A$. Consider the following exhaustive subsubcases:
    * $x = \frac{1}{2}$.
      Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a, b) = (00\frac{1}{2})$. In this way, $D \ni d \triangleq \sim^{\mathfrak{D}} c = (11\frac{1}{2}) \in (D^{\mathcal{A}})^3$.
    * $x = 0$.
      Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a, b) = (000)$. In this way, $D \ni d \triangleq \sim^{\mathfrak{D}} c = (111) \in (D^{\mathcal{A}})^3$.
    * $x = 1$.
      Then, $D \ni c \triangleq \varphi^{\mathfrak{D}}(a, b) = (001)$, in which case $D \ni \sim^{\mathfrak{D}} c = (110)$, and so $D \ni d \triangleq \sim^{\mathfrak{D}}\varphi^{\mathfrak{D}}(c, \sim^{\mathfrak{D}} c) = (111) \in (D^{\mathcal{A}})^3$.
  - $\sim^{\mathfrak{A}}\frac{1}{2} = 1$.
    Then, $D \ni b \triangleq \sim^{\mathfrak{D}} a = (011)$, in which case $D \ni \sim^{\mathfrak{D}} b = (100)$, and so $D \ni d \triangleq \sim^{\mathfrak{D}}\varphi^{\mathfrak{D}}(b, \sim^{\mathfrak{D}} b) = (111) \in (D^{\mathcal{A}})^3$.
  - $\sim^{\mathfrak{A}}\frac{1}{2} = 0$.
    Then, $D \ni b \triangleq \sim^{\mathfrak{D}} a = (010)$, in which case $D \ni \sim^{\mathfrak{D}} b = (101)$, and so $D \ni d \triangleq \sim^{\mathfrak{D}}\varphi^{\mathfrak{D}}(b, \sim^{\mathfrak{D}} b) = (111) \in (D^{\mathcal{A}})^3$.
- $\mathcal{A}$ is not false-singular,
  in which case $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$, while, by Theorem 6.50, 2 forms a subalgebra of $\mathfrak{A}$, and so there is some $\psi \in \mathrm{Fm}^1_\Sigma$ such that $\psi^{\mathfrak{A}}[A] \subseteq 2$. Then, $D \ni b \triangleq \psi^{\mathfrak{D}}(a) \in 2^3$, in which case $D \ni c \triangleq \varphi^{\mathfrak{D}}(b, \sim^{\mathfrak{D}} b) = (3 \times \{0\})$, and so $D \ni d \triangleq \sim^{\mathfrak{D}} c = (3 \times \{1\}) \in (D^{\mathcal{A}})^3$.

Thus, anyway, $d \in ((D^{\mathcal{A}})^3 \cap D) = D^{\mathcal{D}}$, in which case $\mathcal{D}$ is truth-non-empty, and so Lemma 6.54(iii)⇒(i) completes the argument. □

Finally, if C has no theorem, then the purely inferential (and so consistent) IC_{+0} is an extension of C, for $C \subseteq \mathrm{IC}$, in which case $C = C_{+0} \subseteq \mathrm{IC}_{+0}$. And what is more,

IC_{+0}, being inferentially inconsistent, for IC, being an inconsistent $(\infty\setminus 1)$-sublogic of IC_{+0}, is inferentially inconsistent, is not $\sim$-subclassical. Thus, (v)$\Rightarrow$(i) holds. Conversely, assume (i,iii) hold. Consider any consistent extension C' of C. In case $C' = C$, we have $C' = C \subseteq C^{\mathrm{PC}}$. Now, assume $C' \neq C$. If C' was $\sim$-paraconsistent, then so would be its sublogic C, in which case $\mathcal{A}$, being $\sim$-paraconsistent, would be false-singular, and so, by (iii) and Theorem 6.42(iii)$\Rightarrow$(i), C' would be equal to C. Therefore, C' is not $\sim$-paraconsistent. Then, $x_0 \notin T \triangleq C'(\varnothing)$. Moreover, by the structurality of C', $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of C' (in particular, of C), and so is its consistent finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_\Sigma, T \cap \mathrm{Fm}^1_\Sigma\rangle$, in view of (2.23). Then, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.23), $\mathcal{D}$ is a consistent model of C', for $\mathcal{B}$ is so, and so $\mathcal{D}$ is non-$\sim$-paraconsistent, for C' is so, while $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, for $C \subsetneq C'$. And what is more, by (i) and Corollary 3.21(iv)$\Rightarrow$(i), $\mathcal{D}$ is truth-non-empty. Hence, by (2.23), (iii), Lemma 6.48 and Theorem 6.50, a Σ-matrix defining C^{PC} is embeddable into $\mathcal{D}$, in which case $C' \subseteq C^{\mathrm{PC}}$, and so (v) holds, as required. □

Corollary 6.55(i)$\Leftrightarrow$(ii) [resp., Theorem 6.59(i)$\Leftrightarrow$(iv)] provides an effective algebraic criterion of a [three-valued] $\sim$-[sub]classical Σ-logic's having a theorem. In this connection, in view of Corollary 6.57, the instance of the disjunctive K_3/LP without/with theorems and the same underlying algebra of their characteristic matrices, dual to one another, shows that the "[]"-optional reservations in the formulation of Theorem 6.59 are indeed necessary/irrelevant in the "truth-/false-singular" case. This equally concerns the following immediate consequence of Remark 6.46, Corollary 6.57 and Theorem 6.59(i)$\Leftrightarrow$(iii):

Corollary 6.61. *Suppose C is both $\sim$-subclassical and weakly either conjunctive or disjunctive, while $\mathcal{A}$ is [not] false-singular. Then, C has a theorem [iff $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$].*

The following simple example shows that the stipulation of the weak conjunctivity/disjunctivity cannot be omitted in Corollary 6.61 and "Remark 6.46"/"Corollary 6.57", respectively:

Example 6.62. Let $\Sigma \triangleq \{\sim\}$ and $\mathcal{A}$ false-|truth-singular with $\sim^{\mathfrak{A}}\frac{1}{2} = (1|0)$, in which case $[A\setminus]2$ does [not] form a subalgebra of $\mathfrak{A}$, and so, by Theorem 6.50, C is $\sim$-subclassical, while $\langle \sim^{\mathfrak{A}}\frac{1}{2}, \sim^{\mathfrak{A}}(1|0)\rangle = \langle 1|0, 0|1\rangle \notin \theta^{\mathcal{A}} \ni \langle \frac{1}{2}, 1|0\rangle$, in which case $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, whereas M_2 forms as subalgebra of $\mathfrak{A}^2$, in which case, by Lemma 6.56(ii)$\Rightarrow$(iii), $\mathcal{A}$ has no binary semi-conjunction, and so, by Theorem 6.59(i)$\Rightarrow$(iii), C has no theorem. In particular, by Corollary 6.61, C is weakly neither conjunctive nor disjunctive. And what is more, if $h \triangleq h_{+/2}$ would be a homomorphism from $(\mathfrak{A}{\upharpoonright}2)^2$ to $\mathfrak{A}$, then we would have $(1|0) = \sim^{\mathfrak{A}}\frac{1}{2} = \sim^{\mathfrak{A}} h(\langle 1,0\rangle) = h(\sim^{\mathfrak{A}^2}\langle 1,0\rangle) = h(\langle 0,1\rangle) = \frac{1}{2}$. Therefore, $h \notin \hom((\mathfrak{A}{\upharpoonright}2)^2, \mathfrak{A})$. Hence, by Theorem 6.35(i)$\Rightarrow$(v), C is not $\sim$-classical. □

Theorem 6.63. *[Suppose $\mathcal{A}$ is both $\sim$-paraconsistent and weakly conjunctive.] Then, C^{NP} is consistent if[f] C is $\sim$-subclassical.*

Proof. The "if" part is by Remark 2.9(i)**d)** and the consistency of any $\sim$-classical Σ-matrix/-logic. [Conversely, assume C^{NP} is consistent. Then, by Remark 6.46 and Claim 6.60, C has a theorem, in which case, by its structurality, applying the Σ-substitution extending $[x_i/x_0]_{i\in\omega}$ to any theorem of C, we get some $\varphi \in (C(\varnothing) \cap \mathrm{Fm}^1_\Sigma) \subseteq T \triangleq C^{\mathrm{NP}}(\varnothing) \not\ni x_0$. Moreover, by the structurality of C^{NP}, $\langle \mathfrak{Fm}^\omega_\Sigma, T\rangle$ is a model of $C^{[\mathrm{NP}]}$, and so is its consistent truth-non-empty finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^1_\Sigma, T \cap \mathrm{Fm}^1_\Sigma\rangle$, in view of (2.23). Hence, by Lemma

2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, this is both consistent, truth-non-empty and non-$\sim$-paraconsistent, for $\mathcal{B}$ is so, and so $\mathcal{A}$, being $\sim$-paraconsistent, is not a model of the logic of it. In this way, Lemma 6.48 and Theorem 6.50 complete the argument.] $\Box$

The logic IC_{+0} invoked in the proof of Theorem 6.59(v)$\Rightarrow$(i) (held in general) is, though being consistent, is inferentially inconsistent. A proper "inferential" version of this result is then as follows:

Theorem 6.64. *Suppose $\mathcal{A}$ is [not] truth-singular, while C is $\sim$-subclassical. Then, any inferentially consistent extension of C is a sublogic of C^{PC} [iff $\mathcal{A}$ has GC and C has no proper $\sim$-paraconsistent extension iff $\mathcal{A}$ satisfies GC and L_3 does not form a subalgebra of $\mathfrak{A}^2$].*

Proof. [First, the second "iff" part is by Theorem 6.42(i)$\Leftrightarrow$(iv). Likewise, by Theorem 6.42(i)$\Rightarrow$(ii), C has a $\sim$-paraconsistent (and so inferentially consistent) non-$\sim$-subclassical extension, whenever it has a proper $\sim$-paraconsistent one. Now, assume $\mathcal{A}$ does not satisfy GC. Let $\mathcal{B}$ be the submatrix of $\mathcal{A}^2$ generated by $\varnothing \neq \{\langle 1, \frac{1}{2}\rangle\} \subseteq D^{\mathcal{B}}$, for $\mathcal{A}$ is false-singular. Then, by (2.23) and the following claim, the logic of $\mathcal{B}$ is an inferentially consistent (for $\mathcal{B}$ is both consistent and truth-non-empty) extension of C, not being a sublogic of C^{PC}:

Claim 6.65. *Let $\mathcal{B}$ be the submatrix of $\mathcal{A}^2$ generated by $\{\langle 1, \frac{1}{2}\rangle\}$ and C' the logic of $\mathcal{B}$. Suppose $\mathcal{A}$ is false-singular and does not satisfy GC. Then, $(B \setminus D^{\mathcal{B}}) = M_2 \neq \varnothing$, in which case $\sim x_0 \vdash x_0$ is true in $\mathcal{B}$, and so, by (3.11) with $n = 1$ and $m = 0$, $\sim$ is not a subclassical negation for C' (in particular, $C' \neq C$ is not $\sim$-subclassical; cf. Corollary 6.29).*

Proof. Then, $(B \cap \{\langle 0, 0\rangle, \langle 0, \frac{1}{2}\rangle, \langle \frac{1}{2}, 0\rangle\}) = \varnothing$, in which case $\sim^{\mathfrak{A}} \frac{1}{2} = 1$, and so $(B \setminus D^{\mathcal{B}}) = M_2 \neq \varnothing$. On the other hand, for every $a \in M_2$, $\sim^{\mathfrak{B}} a \in M_2$, so the rule $\sim x_0 \vdash x_0$ is true in $\mathcal{B}$, as required. $\Box$

Thus, the first "only if" part holds. Conversely, assume $\mathcal{A}$ has GC, while C has no proper $\sim$-paraconsistent extension.] Consider any inferentially consistent extension C' of C. In case $C' = C$, we have $C' = C \subseteq C^{\mathrm{PC}}$. Now, assume $C' \neq C$. If C' was $\sim$-paraconsistent, then so would be its sublogic C, in which case $\mathcal{A}$, being $\sim$-paraconsistent, would be false-singular, and so, by the []-optional assumption, C' would be equal to C. Therefore, C' is not $\sim$-paraconsistent. Then, $x_1 \notin T \triangleq C'(x_0) \ni x_0$. Moreover, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{2}_{\Sigma}, T \cap \mathrm{Fm}^{2}_{\Sigma}\rangle$, in view of (2.23). Then, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.23), $\mathcal{D}$ is a consistent truth-non-empty model of C', for $\mathcal{B}$ is so, and so $\mathcal{D}$ is non-$\sim$-paraconsistent, for C' is so, while $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, for $C \subsetneq C'$. Hence, by (2.23), (iii), Lemma 6.48 and Theorem 6.50, a Σ-matrix defining C^{PC} is embeddable into $\mathcal{D}$, in which case $C' \subseteq C^{\mathrm{PC}}$. $\Box$

Theorem 6.66. *Suppose $\mathcal{A}$ is either non-$\sim$-paraconsistent (in particular, truth-singular) or weakly conjunctive (viz., C is so). Then, C has a proper inferentially consistent extension iff it is $\sim$-subclassical but not $\sim$-classical, in which case C^{PC} is an extension of any inferentially consistent extension of C.*

Proof. The "if" part is by the inferential consistency of $\sim$-classical Σ-logics. Conversely, consider any proper inferentially consistent extension C' of C, in which case, by Corollary 3.33, C is not $\sim$-classical. Moreover, if C' was $\sim$-paraconsistent, then so would be its sublogic C, in which case this would be weakly conjunctive, and so, by Corollaries 6.29 and 6.47, C' would be equal to C. Therefore, C' is not $\sim$-paraconsistent. Then, $x_1 \notin T \triangleq C'(x_0) \ni x_0$. Moreover, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its consistent truth-non-empty finitely-generated submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{2}_{\Sigma}, T \cap \mathrm{Fm}^{2}_{\Sigma}\rangle$, in view of (2.23). Then, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_{*}(\mathcal{A})^{I}$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.23), $\mathcal{D}$ is a consistent truth-non-empty model of C', for $\mathcal{B}$ is so, and so $\mathcal{D}$ is non-$\sim$-paraconsistent, for C' is so, while $\mathcal{A}$ is not a model of the logic of $\mathcal{D}$, for $C \subsetneq C'$. Hence, by Lemma 6.48, 2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{A}{\restriction}2$ is embeddable into $\mathcal{D}$, whereas, by Theorem 6.50, C is $\sim$-subclassical, in which case C^{PC} is defined by $\mathcal{A}{\restriction}2$, and so, by (2.23), $C' \subseteq C^{\mathrm{PC}}$, as required. □

The initial stipulation in the formulation of Theorem 6.66 cannot be omitted, as it ensues from:

Example 6.67. Let $\mathcal{A}$ be false-singular, $\Sigma \triangleq \{\sim, \top\}$ with nullary $\top$ and $\top^{\mathfrak{A}} \triangleq \sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$, in which case $2 \not\ni \frac{1}{2} = \top^{\mathfrak{A}}$ does not form a subalgebra of $\mathfrak{A}$, while $\langle \sim^{\mathfrak{A}}1, \sim^{\mathfrak{A}}\frac{1}{2}\rangle = \langle 0, \frac{1}{2}\rangle \notin \theta^{\mathcal{A}} \ni \langle 1, \frac{1}{2}\rangle$, in which case $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, whereas $L_4 \not\ni \langle \frac{1}{2}, \frac{1}{2}\rangle = \top^{\mathfrak{A}^2}$ does not form a subalgebra of $\mathfrak{A}^2$, and so, by Theorem[s] 6.35(i)⇒(v) [and 6.50], C is not $\sim$-[sub]classical. On the other hand, $L_3 \ni \langle \frac{1}{2}, \frac{1}{2}\rangle = \top^{\mathfrak{A}^2}$, being closed under $\sim^{\mathfrak{A}^2}$, forms a subalgebra of $\mathfrak{A}^2$, in which case, by Theorem 6.42(i)⇒(iv), C has a proper $\sim$-paraconsistent (and so inferentially consistent) extension, and so, by Theorem 6.66, C is not weakly conjunctive. □

Theorem 6.68. *Suppose $\mathcal{A}$ is false-singular [while, providing C is $\sim$-subclassical, it is either $\sim$-paraconsistent or disjunctive]. Then, C is structurally complete if[f] the following hold:*

(i) *C has a theorem;*
(ii) *C has no proper $\sim$-paraconsistent extension;*
(iii) *$\mathcal{A}$ satisfies GC;*
(iv) *$\mathcal{A}$ satisfies DGC;*
(v) *L_4 does not form a subalgebra of $\mathfrak{A}^2$;*
(vi) *C is $\sim$-subclassical iff it is $\sim$-classical;*

in which case, providing C is not $\sim$-classical, any three-valued expansion of it is structurally complete. In particular, providing C is $\sim$-paraconsistent, it is structurally complete iff $\mathcal{A}$ satisfies both GC and DGC, while C is both maximally $\sim$-paraconsistent and neither $\sim$-subclassical nor purely-inferential, whereas L_4 does not form a subalgebra of $\mathfrak{A}^2$.

Proof. First, assume (i–vi) hold. Then, in case C is $\sim$-classical, by (i) and Corollary 3.33, it is structurally complete. Now, assume C is not $\sim$-classical, and so is not $\sim$-subclassical, in view of (vi). Let C' be any extension of C such that $T \triangleq C'(\varnothing) = C(\varnothing) \not\ni x_0$, by the consistency of $\mathcal{A}$, and so of C. Then, by (2.23), (i) and the structurality of C', $\mathcal{B} \triangleq \langle \mathfrak{Fm}^{1}_{\Sigma}, T \cap \mathrm{Fm}^{1}_{\Sigma}\rangle$ is a finitely-generated consistent truth-non-empty model of C' (in particular, of C), in which case, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_{*}(\mathcal{A})^{I}$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.23), $\mathcal{D}$ is a consistent truth-non-empty model of C', for $\mathcal{B}$ is so. Consider the following complementary cases:

- $\mathcal{D}$ is $\sim$-paraconsistent.
  Then, by (2.23), (ii), Lemma 6.41 and Theorem 6.42(i)⇒(iii), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so.

- $\mathcal{D}$ is not $\sim$-paraconsistent.
  Then, as C is not $\sim$-subclassical, by (iii-v), Lemma 6.48 and Theorem 6.50, $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so.

Thus, anyway, $\mathcal{A} \in \mathrm{Mod}(C')$, in which case C', being an extension of C, is equal to C, and so C is structurally complete. [Conversely, assume either of (i-vi) does not hold. Consider respective cases:

(i) does not hold.
Then, by Remark 2.5, C, being inferentially consistent, for $\mathcal{A}$ is both consistent and truth-non-empty, is not structurally complete.

(ii) does not hold.
Then, by Theorem 6.45, $C_{\frac{1}{2}}$ is a proper extension of C, while $\Delta_A \in \hom^{\mathrm{S}}(\mathcal{A}_{\frac{1}{2}}, \mathcal{A})$, in which case, by (2.24), $C_{\frac{1}{2}}(\varnothing) = C(\varnothing)$, and so C is not structurally complete.

(iii) does not hold.
Let $\mathcal{B}'$ be the submatrix of $\mathcal{A}^2$ generated by $\{\langle 1, \frac{1}{2}\rangle\}$. Then, by (2.23) and Claim 6.65, the logic C' of $\mathcal{B}'$ is a proper extension of C, while $(\pi_1 \restriction B') \in \hom^{\mathrm{S}}(\mathcal{B}', \mathcal{A})$, for $\pi_1[M_2] = 2$, in which case, by (2.24), $C'(\varnothing) = C(\varnothing)$, and so C is not structurally complete.

(iv) does not hold.
Let $\mathcal{B}'$ be the submatrix of $\mathcal{A}^2$ generated by $M_2 \cup \{\langle 1, \frac{1}{2}\rangle\}$ and C' the logic of $\mathcal{B}'$. Then, as $\langle 0, 0\rangle \notin B'$, while $\sim^{\mathfrak{A}} 1 = 0$, $\sim^{\mathfrak{A}} \frac{1}{2} \neq 0$, in which case $\mathcal{A}$ is $\sim$-paraconsistent, and so is C. Moreover, as $\langle \frac{1}{2}, \frac{1}{2}\rangle \notin B'$, $\mathcal{B}'$ is non-$\sim$-paraconsistent, and so is C', in which case, by (2.23), C' is a proper extension of C. Moreover, $(\pi_1 \restriction B') \in \hom^{\mathrm{S}}(\mathcal{B}', \mathcal{A})$, for $\pi_1[M_2] = 2$, in which case, by (2.24), $C'(\varnothing) = C(\varnothing)$, and so C is not structurally complete.

(v) does not hold.
Let $\mathcal{B}' \triangleq (\mathcal{A}^2 \restriction L_4)$ and C' the logic of $\mathcal{B}'$. Then, as $\langle 0, 0\rangle \notin L_4$, while $\sim^{\mathfrak{A}} 1 = 0$, $\sim^{\mathfrak{A}} \frac{1}{2} \neq 0$, in which case $\mathcal{A}$ is $\sim$-paraconsistent, and so is C. Moreover, as $\langle \frac{1}{2}, \frac{1}{2}\rangle \notin L_4$, $\mathcal{B}'$ is non-$\sim$-paraconsistent, and so is C', in which case, by (2.23), C' is a proper extension of C. Moreover, $(\pi_1 \restriction L_4) \in \hom^{\mathrm{S}}(\mathcal{B}', \mathcal{A})$, for $\pi_1[L_4] = A$, in which case, by (2.24), $C'(\varnothing) = C(\varnothing)$, and so C is not structurally complete.

(vi) does not hold,
in which case C is $\sim$-subclassical but not $\sim$-classical. Let $\mathcal{B}' \triangleq \mathcal{A}_{\mathrm{PC}} \in \mathrm{Mod}(C)$. Then, $\mathcal{D} \triangleq (\mathcal{A} \times \mathcal{B}')$ is a model of C, in which case the logic C' of $\mathcal{D}$ is an extension of C, and so, as $(\pi_0 \restriction D) \in \hom^{\mathrm{S}}(\mathcal{D}, \mathcal{A})$, by (2.24), we have $C'(\varnothing) = C(\varnothing)$. For proving the fact that $C' \neq C$, consider the following complementary cases:

- $\mathcal{A}$ is $\sim$-paraconsistent,
  and so is C. Then, by Remark 2.9(i)**d)**,(iii), C' is not $\sim$-paraconsistent, for $\mathcal{B}'$, being $\sim$-negative, is so, and so $C' \neq C$.
- $\mathcal{A}$ is not $\sim$-paraconsistent,
  in which case it is $\sim$-negative. Then, C, being both $\sim$-subclassical and non-$\sim$-paraconsistent, is $\veebar$-disjunctive, and so is $\mathcal{A}$, in view of Lemma 6.30, in which case, by Remark 2.9(i)**c)**, it is implicative, while $\mathcal{D}$ is weakly $\veebar$-disjunctive, whereas, by Corollary 6.51, 2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{B}' = (\mathcal{A} \restriction 2)$. Moreover, by Corollary 6.36, $\mathcal{A}$ is hereditarily simple, and so is $\mathcal{D}$, by Lemmas 3.10, 3.13, Theorem 3.11(i)$\Leftrightarrow$(iii) and Remark 5.13(iv). We prove that $C' \neq C$ by contradiction. For suppose $C' = C$, in which case $\mathcal{A}$ is a finite model of C', and so, by Corollary 3.20 and Remark 2.8(ii), there is some

$h \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{A}, \mathcal{D})$. Then, $g \triangleq ((\pi_1 \restriction D) \circ h) \in \hom^{\mathrm{S}}(\mathcal{A}, \mathcal{B}')$, in which case, as $D^{\mathcal{A}} = \{1, \frac{1}{2}\}$, we have $g(1) = 1 = g(\frac{1}{2})$, and so g is not injective, while $0 = \sim^{\mathfrak{B}} 1 = \sim^{\mathfrak{B}} g(1) = g(\sim^{\mathfrak{A}} 1) = g(0)$. Hence, g is strict. This contradicts to Remark 2.8(ii). Thus, $C' \neq C$.

Thus, anyway, $C' \neq C$, in which case C is not structurally complete.

Thus, in any case, C is not structurally complete.]

Finally, as expansions of $\mathcal{A}/C$ inherit (iii-v)/"both (i) and absence of $\sim$-classical models", respectively, Corollary 6.40 and Theorem 6.42 complete the proof. □

Remark 2.5 and Theorem 6.68 inevitably raise the problem of finding the structural completion of C, whenever it is both $\sim$-paraconsistent and $\sim$-subclassical but not purely-inferential.

Lemma 6.69. *Let $i \in 2$, $\mathcal{K}'_{3,i}$ the submatrix of $\mathcal{A}^2$ generated by $K_{3,i} \triangleq (\Delta_2 \cup \{\langle \frac{1}{2}, i\rangle\})$. Suppose 2 forms a subalgebra of $\mathfrak{A}$, in which case C is $\sim$-subclassical, C^{PC} being defined by $\mathcal{A} \restriction 2$; cf. Theorem 6.50. Then, the following are equivalent:*

(i) $\langle 0, 1\rangle \in K'_{3,i}$;
(ii) $\langle 1, 0\rangle \in K'_{3,i}$;
(iii) $M_2 \subseteq K'_{3,i}$;
(iv) $(M_2 \cap K'_{3,i}) \neq \varnothing$;
(v) $K'_{3,i} \nsubseteq K_4 \triangleq (\bigcup_{j \in 2} K_{3,j})$;
(vi) *neither $K_{3,i}$ nor K_4 forms a subalgebra of $\mathfrak{A}^2$.*

Moreover, providing $\mathcal{A}$ is ([both $\bar{\wedge}$-conjunctive and] $\veebar$-disjunctive as well as) false-singular {more specifically, $\sim$-paraconsistent}, **a)**$\Leftrightarrow$**b)**$\Rightarrow$ ($\Leftrightarrow$)**c)**$\Rightarrow$ {$\Leftrightarrow$}**d)**$\Leftarrow$ ([$\Leftrightarrow$])**e)**$\langle\Rightarrow$**b)**$\rangle$, *where:*

a) *C^{PC} is a proper axiomatic extension of C;*
b) *$C^{\mathrm{PC}}(\varnothing) \neq C(\varnothing)$;*
c) *$\langle 0, 1\rangle \in \bigcap_{j \in 2} K'_{3,j}$ $\langle$while C is not $\sim$-classical$\rangle$;*
d) *$\langle 0, 1\rangle \in K'_{3,0}$ $\langle$while C is not $\sim$-classical$\rangle$;*
e) *$\mathcal{A}$ is implicative $\langle$while C is not $\sim$-classical$\rangle$.*

In particular, the non-$\langle\rangle$-optional versions of **a)**–**e)** *are equivalent, whenever C is both conjunctive and disjunctive as well as both $\sim$-paraconsistent and $\sim$-subclassical.*

Proof. First, (i)$\Leftrightarrow$(ii) is by the fact that $\sim^{\mathfrak{A}} j = (1 - j)$, for all $j \in 2$, while (iii/iv) is the conjunction/disjunction of (i) and (ii). Next, (iii)$\Rightarrow$(v) is by the fact that $M_2 \nsubseteq K_4$. Further, (v)$\Rightarrow$(vi) is by the fact that $K_{3,i} \subseteq K_4$. The converse is by the fact that $K_4 = (K_{3,i} \cup \{\langle \frac{1}{2}, 1 - i\rangle\})$, while $K_{3,i} \subseteq K'_{3,i}$. Furthermore, (v)$\Rightarrow$(iv) is by the fact that $K_4 = ((A \times 2) \setminus M_2)$, while $K'_{3,i} \subseteq (A \times 2)$, for $\pi_1[K_{3,i}] = 2$ forms a subalgebra of $\mathfrak{A}$.

Now, suppose $\mathcal{A}$ is [both $\bar{\wedge}$-conjunctive and] $\veebar$-disjunctive (in which case $\mathcal{A} \restriction 2$ is so; cf. Remark 2.9(ii)) as well as false-singular {more specifically, $\sim$-paraconsistent}.

First, **b)** is a particular case of **a)**. Conversely, assume **b)** holds. Then, $C^{\mathrm{PC}}(\varnothing) \nsubseteq C(\varnothing)$, for $C \subseteq C^{\mathrm{PC}}$, in which case there is some $\varphi \in (C^{\mathrm{PC}}(\varnothing) \setminus C(\varnothing)) \neq \varnothing$, and so φ is true in $\mathcal{A} \restriction 2$ but is not true in $\mathcal{A}$. On the other hand, $\mathcal{A} \restriction 2$ is the only proper consistent submatrix of $\mathcal{A}$. Hence, by Corollary 2.14, C^{PC} is the axiomatic extension of C relatively axiomatized by φ. Thus, **a)** holds.

Next, **d)** is a particular case of **c)**. {Conversely, assume $\langle 0, 1\rangle \in K'_{3,0}$. Consider the following complementary cases:

- $\sim^{\mathfrak{A}}\frac{1}{2}=\frac{1}{2}$.
  Then, $\langle\frac{1}{2},0\rangle=\sim^{\mathfrak{A}^2}\langle\frac{1}{2},1\rangle\in K'_{3,1}$, for $K'_{3,1}\supseteq K_{3,1}\ni\langle\frac{1}{2},1\rangle$ forms a subalgebra of $\mathfrak{A}^2$, in which case $K_{3,0}=(\Delta_2\cup\{\langle\frac{1}{2},0\rangle\})\subseteq K'_{3,1}$, for $\Delta_2\subseteq K_{3,1}\subseteq K'_{3,1}$, and so, $K'_{3,1}$, forming a subalgebra of $\mathfrak{A}^2$, includes $K'_{3,0}\ni\langle 0,1\rangle$.
- $\sim^{\mathfrak{A}}\frac{1}{2}\neq\frac{1}{2}$, in which case $\sim^{\mathfrak{A}}\frac{1}{2}=1$,
  for $\mathcal{A}$ is $\sim$-paraconsistent, and so $\langle 0,1\rangle=\sim^{\mathfrak{A}^2}\sim^{\mathfrak{A}^2}\langle\frac{1}{2},1\rangle\in K'_{3,1}$, for $K'_{3,1}\supseteq K_{3,1}\ni\langle\frac{1}{2},1\rangle$ forms a subalgebra of $\mathfrak{A}^2$.

Thus, in any case, $\langle 0,1\rangle\in\bigcap_{j\in 2}K'_{3,j}$, and so **d)**$\Rightarrow$**c)** holds.}

(Further, assume $\langle 0,1\rangle\in\bigcap_{j\in 2}K'_{3,j}$. Then, there is some $\bar{\phi}\in(\mathrm{Fm}^3_\Sigma)^2$ such that, for each $j\in 2$, $\phi^{\mathfrak{A}}_j(0,\frac{1}{2},1)=0$ and $\phi^{\mathfrak{A}}_j(0,j,1)=1$. Moreover, by Remark 2.9(i)**d)**, $\varphi\triangleq(2.18)\in(C^{\mathrm{PC}}(\varnothing)\cap\mathrm{Fm}^1_\Sigma)$. Set $\psi\triangleq(\veebar\bar{\phi}[x_0/{\sim}\varphi,x_2/\varphi])\in\mathrm{Fm}^2_\Sigma$. Then, since both $\mathcal{A}$ and $\mathcal{A}{\restriction}2$ are $\veebar$-disjunctive as well as false-singular, while the latter is also both $\sim$-negative and truth-singular, we have, for all $k\in 2$, $\psi^{\mathfrak{A}}(k,\frac{1}{2})=0$ as well as $\psi^{\mathfrak{A}}(k,l)=1$, for all $l\in 2$, in which case ψ is not true in $\mathcal{A}$ under $[x_0/k,x_1/\frac{1}{2}]$ but is true in $\mathcal{A}{\restriction}2$, and so $\psi\in(C^{\mathrm{PC}}(\varnothing)\setminus C(\varnothing))$. Thus, **c)**$\Rightarrow$**b)** holds.) Conversely, if $\langle 0,1\rangle\notin K'_{3,j}$, for some $j\in 2$, then $(\pi_1{\restriction}K'_{3,j})\in\hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}'_{3,j},\mathcal{A}{\restriction}2)$, because $\pi_1[K_{3,j}]=2$ forms a subalgebra of $\mathfrak{A}$, in which case, by (2.23), C^{PC} is defined by $\mathcal{K}'_{3,j}$, and so, by (2.24), $C^{\mathrm{PC}}(\varnothing)=C(\varnothing)$, for $(\pi_0{\restriction}K'_{3,j})\in\hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}'_{3,j},\mathcal{A})$, because $\pi_0[K_{3,j}]=A$. $\langle$Likewise, if C is $\sim$-classical, then, by Lemma 3.32, $C^{\mathrm{PC}}=C$, for $C\subseteq C^{\mathrm{PC}}$.$\rangle$ Thus, **b)**$\Rightarrow$**c)** holds.

$\langle$Furthermore, if **e)** holds, then, as $C\subseteq C^{\mathrm{PC}}$, **b)** is by Remark 2.9(ii) and Lemma 3.28.$\rangle$

Finally, assume $\mathcal{A}$ is $\sqsupset$-implicative. Then, as $0\notin D^{\mathcal{A}}$, we have both $(\frac{1}{2}\sqsupset^{\mathfrak{A}}0)=0$, for $\mathcal{A}$ is false-singular, and $(0\sqsupset^{\mathfrak{A}}0)=1$, for 2 forms a subalgebra of $\mathfrak{A}$. Therefore, since $K'_{3,0}\supseteq K_{3,0}\ni\langle 0/\frac{1}{2},0\rangle$ forms a subalgebra of $\mathfrak{A}^2$, we get $\langle 0,1\rangle=(\langle\frac{1}{2},0\rangle\sqsupset^{\mathfrak{A}^2}\langle 0,0\rangle)\in K'_{3,0}$. Thus, **e)**$\Rightarrow$**d)** holds. ([Conversely, assume $\langle 0,1\rangle\in K'_{3,0}$. Then, there is some $\phi\in\mathrm{Fm}^3_\Sigma$ such that $\phi^{\mathfrak{A}}(\frac{1}{2},0,1)=0$, while $\phi^{\mathfrak{A}}(0,0,1)=1$, in which case $\psi\triangleq(\phi[x_2/{\sim}x_1])\in\mathrm{Fm}^2_\Sigma$, while $\psi^{\mathfrak{A}}(\frac{1}{2},0)=0$, whereas $\psi^{\mathfrak{A}}(0,0)=1$, and so $\varphi\triangleq(\psi\barwedge{\sim}x_0)\in\mathrm{Fm}^2_\Sigma$, while $\varphi^{\mathfrak{A}}(a,0)=(1-\chi^{\mathcal{A}}(a))$, for all $a\in A$, for $\mathcal{A}$ is both $\barwedge$-conjunctive and false-singular. In this way, by the following claim, $\mathcal{A}$, being $\veebar$-disjunctive, is implicative:

Claim 6.70. *Let $\mathcal{N}'_2$ be the submatrix of $\mathcal{A}^3$ generated by $N_2\triangleq\{\langle 0,1,\frac{1}{2}\rangle,\langle 0,0,0\rangle\}$. Suppose $\mathcal{A}$ is false-singular, while 2 forms a subalgebra of $\mathfrak{A}$. Then, $\mathcal{A}$ is implicative iff it is disjunctive, while $\langle 1,0,0\rangle\in N'_2$.*

Proof. First, if $\mathcal{A}$ is $\sqsupset$-implicative, then it is $\uplus_{\sqsupset}$-disjunctive, while $N'_2\ni(\langle 0,1,\frac{1}{2}\rangle\sqsupset^{\mathfrak{A}^3}\langle 0,0,0\rangle)=\langle 1,0,0\rangle$, for $N'_2\supseteq N_2$ forms a subalgebra of $\mathfrak{A}^3$, while 2 forms a subalgebra of $\mathfrak{A}$, whereas $\mathcal{A}$ is false-singular. Conversely, assume $\mathcal{A}$ is $\veebar$-disjunctive, while, $\langle 1,0,0\rangle\in N'_2$, in which case there is some $\phi\in\mathrm{Fm}^2_\Sigma$ such that $\phi^{\mathfrak{A}}(a,0)=(1-\chi^{\mathcal{A}}(a))$, for all $a\in A$, and so $\psi\triangleq(\phi\veebar x_1)\in\mathrm{Fm}^2_\Sigma$, while $\mathcal{A}$, being false-singular, is ψ-implicative. □

Thus, **d)**$\Rightarrow$**e)** holds.])

In this way, Remark 2.9(i)**d)**, Lemmas 6.30, 6.31 and Corollary 6.51 complete the argument. □

By Corollaries 3.33, 6.51, 6.61, Lemmas 3.32, 6.69 and Remark 2.9(i)**d)**, we immediately get:

Corollary 6.71. *Suppose C is [both conjunctive and] disjunctive as well as $\sim$-subclassical, while $\mathcal{A}$ is false-singular (more specifically, C is $\sim$-paraconsistent).*

Then, C^{PC} is the structural completion of C iff [either C is $\sim$-classical or] either K_4 or $K_{3,i}$, for some $i \in (2(\cap 1)[\cap 1])$, forms a subalgebra of $\mathfrak{A}^2$ [if and] only if C is either $\sim$-classical or non-implicative. In particular, providing C is $\sim$-paraconsistent, C^{PC} is the structural completion of it iff either K_4 or $K_{3,0}$ forms a subalgebra of $\mathfrak{A}^2$ [if and] only if it is not implicative.

The opposite case is analyzed in Subsubsection 6.2.2 below within the framework of $\sim$-paraconsistent three-valued Σ-logics with subclassical negation $\sim$ as well as lattice conjunction and disjunction. On the other hand, the []-optional stipulation of conjunctivity cannot be omitted in the formulations of Lemma 6.69 and Corollary 6.71, even if C is $\sim$-paraconsistent, in view of:

Example 6.72. Let $\Sigma \triangleq \{\diamond, \vee, \sim\}$ with binary $\sim$ and $\mathcal{A}$ false-singular with $\sim^{\mathfrak{A}}\frac{1}{2} \triangleq \frac{1}{2}$ and

$$(a(\vee|\diamond)^{\mathfrak{A}}b) \triangleq \begin{cases} \frac{1}{2}|0 & \text{if } \frac{1}{2} \in \{a, b\}, \\ \max(a,b)|1 & \text{otherwise,} \end{cases}$$

for all $a, b \in A$. Then, 2 forms a subalgebra of $\mathfrak{A}$, while $\mathcal{A}$ is both $\sim$-paraconsistent and $\vee$-disjunctive, whereas $\langle 0,1\rangle = (\langle \frac{1}{2}, 0\rangle \diamond^{\mathfrak{A}^2} \langle 0,0\rangle) \in K'_{3,0}$, for $K'_{3,0} \supseteq K_{3,0} \supseteq \{\langle \frac{1}{2}, 0\rangle, \langle 0,0\rangle\}$ forms a subalgebra of $\mathfrak{A}^2$. On the other hand, $((2^2 \times \{\frac{1}{2}\}) \cup (\Delta_2 \times 2)) \supseteq N_2$ forms a subalgebra of $\mathfrak{A}^3$ but does not contain $\langle 1,0,0\rangle$, for $1 \neq 0 \neq \frac{1}{2}$, in which case, by Claim 6.70, $\mathcal{A}$ is not implicative, and so is not conjunctive, in view of Lemma 6.69**d)**$\Rightarrow$**e)**. □

Remark 6.73. Let φ be a binary semi-conjunction for $\mathcal{A}$. Then, $\varphi^{\mathfrak{A}^2}(\langle 0,1\rangle, \langle 1,0\rangle) = \langle 0,0\rangle \in \Delta_A$, so $\mathcal{A}$ satisfies DGC. □

Remark 6.74. Suppose $\mathcal{A}$ is both false-singular and weakly $\bar{\wedge}$-conjunctive (viz., C is so). Then, as 0 is the only non-distinguished value of $\mathcal{A}$, we have $(0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} a) = 0 = (a \mathbin{\bar{\wedge}^{\mathfrak{A}}} 0)$, for all $a \in A$, in which case we get $(\langle 0,a\rangle \mathbin{\bar{\wedge}^{\mathfrak{A}^2}} \langle a,0\rangle) = \langle 0,0\rangle \notin L_4 \supseteq \{\langle 0,\frac{1}{2}\rangle, \langle \frac{1}{2},0\rangle\}$, and so, in particular, L_4 does not form a subalgebra of $\mathfrak{A}^2$, while, in case $\sim^{\mathfrak{A}}\frac{1}{2} = 1$, we have $\langle 0,0\rangle = (\sim^{\mathfrak{A}^2}\langle 1,\frac{1}{2}\rangle \mathbin{\bar{\wedge}^{\mathfrak{A}^2}} \sim^{\mathfrak{A}^2}\sim^{\mathfrak{A}^2}\langle 1,\frac{1}{2}\rangle)$, whereas, otherwise, we have $\sim^{\mathfrak{A}^2}\langle 1,\frac{1}{2}\rangle \in \{\langle 0,0\rangle, \langle 0,\frac{1}{2}\rangle\}$. Thus, in addition, $\mathcal{A}$ satisfies GC. □

Combining Theorems 6.42(iii)$\Rightarrow$(i), 6.59(iii)$\Rightarrow$(i), 6.68 with Remarks 6.46, 6.73, 6.74 and Corollary 3.33, we immediately get:

Corollary 6.75. *Suppose $\mathcal{A}$ is false-singular (in particular, $\sim$-paraconsistent) and weakly conjunctive. Then, C is structurally complete iff it is either $\sim$-classical or non-$\sim$-subclassical.*

Further, $\mathcal{A}$ is said to be *classically-hereditary*, provided 2 forms a subalgebra of $\mathfrak{A}$. Likewise, $\mathcal{A}$ is said to be *classically-valued*, provided, for each $\varsigma \in \Sigma$, $(\operatorname{img} \varsigma^{\mathfrak{A}}) \subseteq 2$, in which case it is classically-hereditary.

Remark 6.76. Suppose $\mathcal{A}$ is both classically-valued and $\sqsupset$-implicative. Then, as $1 \in D^{\mathcal{A}} \not\ni 0$, we have $(a \sqsupset^{\mathfrak{A}} a) = 1$, for all $a \in A$, in which case, as $\sim^{\mathfrak{A}}1 = 0$, $\mathcal{A}$ is $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \sqsupset \sim(x_0 \supset x_0))$, and so $\uplus^{\sim}_{\sqsupset}$-conjunctive, in view of Remark 2.9(i)**a)**. □

Combining Remarks 2.9(i)**d)** and 6.76 with Corollaries 6.51 and 6.75, we also have:

Corollary 6.77. *Let $c \notin \Sigma$ be a nullary connective, $\Sigma' \triangleq (\Sigma \cup \{c\})$, $\mathcal{A}'$ the Σ'-expansion of $\mathcal{A}$ with $c^{\mathfrak{A}'} \triangleq \frac{1}{2}$ and C' the logic of $\mathcal{A}'$. Suppose $\mathcal{A}$ is $\sim$-paraconsistent as well as both classically-hereditary and weakly conjunctive (in particular, both implicative and classically-valued). Then, C' is structurally complete, while C is not so, whereas both C and C' are maximally $\sim$-paraconsistent.*

This covers, in particular, both LP, LA, HZ (remark that this is $\vee^{\sim}$-conjunctive) — as non-classically-valued conjunctive classically-hereditary instances — and P^1 — as a term-wise definitionally minimal classically-valued implicative instance — as well as their bounded expansions by classical constants $\bot$ and $\top$ interpreted as 0 and 1, respectively. (In this connection, recall that the fact that LP is "maximally $\sim$-paraconsistent"/"not structurally complete" has been due to [17]/[20], respectively, proved *ad hoc* therein.) Thus, in view of Corollaries 6.29, 6.77 and Theorem 6.68, any $\sim$-paraconsistent three-valued $\sim$-paraconsistent Σ-logic with subclassical negation $\sim$ is maximally so, whenever it is structurally complete, while the converse does not, generally speaking, hold, whereas the structural completeness of such a logic subsumes absence of its $\sim$-classical extensions. On the other hand, the situation with paracompleteness is quite different. First, we have:

Lemma 6.78. *Suppose C is maximally $(\veebar,\sim)$-paracomplete. Then, it is structurally complete.*

Proof. In that case, any extension C' of C such that $C'(\varnothing) = C(\varnothing)$ is $(\veebar,\sim)$-paracomplete as well, and so equal to C, as required. □

Lemma 6.79. *Let $\mathcal{K}'_3$ be the submatrix of $\mathcal{A}^2$ generated by $K_3 \triangleq K_{3,1}$ and C' the logic of $\mathcal{K}'_3$. Suppose C is both $\veebar$-disjunctive and $(\veebar,\sim)$-paracomplete (viz. $\mathcal{A}$ is so; cf. Lemma 6.30) as well as $\sim$-subclassical. Then, C' is an extension of C such that $C'(\varnothing) = C(\varnothing)$, in which case it is $(\veebar,\sim)$-paracomplete, and so inferentially $(\veebar,\sim)$-paracomplete (in particular, C' is a proper sublogic of C^{PC}). Moreover, (i)⇒ [⇔](ii)⇔(iii)⇔(iv)⇔(v), where:*

- (i) *$\mathcal{A}$ is implicative;*
- (ii) *$\langle 1,0\rangle \in K'_3$ [and C has a theorem];*
- (iii) *$K'_3 \nsubseteq K_4$ [and C has a theorem];*
- (iv) *[both] neither K_3 nor K_4 forms a subalgebra of $\mathfrak{A}^2$ [and C has a theorem];*
- (v) *$C \neq C'$ [has a theorem].*

Proof. In that case, $\mathcal{A}$ is truth-singular, while, by Remark 2.9(i)**d)**, C is not $\sim$-classical, and so, by Corollary 6.51, 2 forms a subalgebra of $\mathfrak{A}$, while, by (2.23), C' is an extension of C. And what is more, as $\pi_0[K_3] = A$, $(\pi_0\restriction K'_3) \in \hom^{\mathrm{S}}(\mathcal{K}'_3,\mathcal{A})$, in which case, by (2.24), $C'(\varnothing) = C(\varnothing)$, and so C' (viz., $\mathcal{K}'_3$) is $(\veebar,\sim)$-paracomplete. Hence, as $\mathcal{K}'_3$ is truth-non-empty, for $\langle 1,1\rangle \in K_3$, it (viz., C') is inferentially $(\veebar,\sim)$-paracomplete, in which case C' is inferentially consistent, and so, by Remark 2.9(i)**d)** and Theorem 6.64, is a proper sublogic of C^{PC}.

Next, assume $\mathcal{A}$ is $\sqsupset$-implicative, in which case, since $D^{\mathcal{A}} = \{1\}$, $(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = 1$ and, as 2 forms a subalgebra of $\mathfrak{A}$, $(1 \sqsupset^{\mathfrak{A}} 0) = 0$, and so $\langle 1,0\rangle = (\langle\frac{1}{2},1\rangle \sqsupset^{\mathfrak{A}^2} \langle 0,0\rangle) \in K'_3$, for $\{\langle\frac{1}{2},1\rangle,\langle 0,0\rangle\} \subseteq K_3 \subseteq K'_3$. Thus, (i)⇒(ii) holds [in view of (2.12)].

[Conversely, assume (ii) holds, in which case, by the following claim, there is some $\phi \in \mathrm{Fm}^1_\Sigma$ such that $\phi^{\mathfrak{A}}(\frac{1}{2}) = 1$, while $\phi^{\mathfrak{A}}(1) = 0$, there is some $\phi \in \mathrm{Fm}^1_\Sigma$ such that $\phi^{\mathfrak{A}}(\frac{1}{2}) = 1$, while $\phi^{\mathfrak{A}}(1) = 0$:

Claim 6.80. *Suppose $\mathcal{A}$ is truth-singular, while C has a theorem, whereas $\langle 1,0\rangle \in K'_3$. Then, there is some $\phi \in \mathrm{Fm}^1_\Sigma$ such that $\phi^{\mathfrak{A}}(\frac{1}{2}) = 1$, while $\phi^{\mathfrak{A}}(1) = 0$.*

Proof. In that case, there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2},1,0) = 1$, while $\varphi^{\mathfrak{A}}(1,1,0) = 0$, and so we have $\psi \triangleq \varphi[x_2/{\sim}x_1] \in \mathrm{Fm}^2_\Sigma$ such that $\psi^{\mathfrak{A}}(\frac{1}{2},1) = 1$, while $\psi^{\mathfrak{A}}(1,1) = 0$. Take any $\zeta \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing)) \neq \varnothing$, in view of the structurality of C. Then, $\phi \triangleq \psi[x_1/\zeta] \in \mathrm{Fm}^1_\Sigma$, while $\phi^{\mathfrak{A}}(\frac{1}{2}) = 1$, whereas $\phi^{\mathfrak{A}}(1) = 0$, for $\mathcal{A}$ is truth-singular. □

Then, $\xi \triangleq (\phi \veebar {\sim}x_0) \in \mathrm{Fm}^1_\Sigma$, in which case $\mathcal{A}$, being truth-singular and $\veebar$-disjunctive, is ξ-negative, and so (i) is by Remark 2.9(i)**c)**.]

Further, (ii)⇔(iii)⇔(iv) is by Lemma 6.69(ii)⇔(v)⇔(vi) with $i = 1$.

Finally, assume (ii) holds. We prove that $C' \neq C$, by contradiction. For suppose $C' = C$, in which case $\mathcal{A}$ is a finite consistent truth-non-empty $\veebar$-disjunctive simple (in view of Theorem 6.35(iv)⇒(i)) model of $C' \supseteq C$, being, in its turn, weakly $\veebar$-disjunctive, and so is $\mathcal{K}'_3$. Then, by Corollary 3.20 and Remark 2.8, there is some submatrix $\mathcal{D}$ of $\mathcal{K}'_3$, being a strict surjective homomorphic counter-image of $\mathcal{A}$, in which case, by (2.23) and Remark 2.9(ii), it is both truth-non-empty, $(\veebar, \sim)$-paracomplete and $\veebar$-disjunctive, for $\mathcal{A}$ is so, and so $D^{\mathcal{D}} = \{\langle 1,1\rangle\}$, while there is some $a \in D$ such that $D \ni b \triangleq (a \veebar^{\mathfrak{A}^2} \sim^{\mathfrak{A}^2} a) \notin D^{\mathcal{D}} = \{\langle 1,1\rangle\}$. On the other hand, since $\pi_1[K_3] = 2$ forms a subalgebra of $\mathfrak{A}$, in which case $\pi_1[D] \subseteq \pi_1[K'_3] \subseteq 2$, by the truth-singularity and $\veebar$-disjunctivity of $\mathcal{A}$, we have $\pi_1(b) = 1$, in which case $\pi_0(b) \neq 1$, and so we have the following two exhaustive cases:

- $\pi_0(b) = \frac{1}{2}$.
  Then, as $\langle 0,0\rangle = \sim^{\mathfrak{A}^2}\langle 1,1\rangle \in D$, we have $K_3 \subseteq D$, in which case, by (ii), we get $\langle 1,0\rangle \in D$, and so $\langle 0,1\rangle = \sim^{\mathfrak{A}^2}\langle 1,0\rangle \in D$.
- $\pi_0(b) = 0$.
  Then, we also have $\langle 1,0\rangle = \sim^{\mathfrak{A}^2}\langle 0,1\rangle \in D$.

Thus, anyway, $M_2 \subseteq (D \setminus D^{\mathcal{D}})$, while, by the $\veebar$-disjunctivity of $\mathcal{A}$, $(\langle 0,1\rangle \veebar^{\mathfrak{A}^2} \langle 1,0\rangle) = \langle 1,1\rangle \in D^{\mathcal{D}}$. This contradicts to the $\veebar$-disjunctivity of $\mathcal{D}$. Thus, (v) holds. Conversely, assume $\langle 1,0\rangle \notin K'_3$, in which case $(\pi_0 \restriction B) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{K}'_3, \mathcal{A})$, and so $C' = C$, by (2.23), as required. □

Lemma 6.81. *Suppose C is weakly $\veebar$-disjunctive (viz. $\mathcal{A}$ is so), while, providing C is $\sim$-subclassical, either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$. Then, C has no proper inferentially $(\veebar, \sim)$-paracomplete extension.*

Proof. Let C' be an inferentially $(\veebar, \sim)$-paracomplete (and so inferentially consistent) extension of C, in which case $(x_1 \veebar \sim x_1) \notin T \triangleq C'(x_0) \ni x_0$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its $(\veebar, \sim)$-paracomplete truth-non-empty finitely-generated submatrix $\mathcal{D} \triangleq \langle \mathfrak{Fm}^2_{\Sigma}, \mathrm{Fm}^2_{\Sigma} \cap T\rangle$, in view of (2.23), whereas C is [inferentially] $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so), in which case, since $\mathcal{A}$ is weakly $\veebar$-disjunctive and $1 \in D^{\mathcal{A}}$, and so $((1/0) \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}}(1/0)) = ((1/0) \veebar^{\mathfrak{A}} (0/1)) \in D^{\mathcal{A}}$, we have $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) \notin D^{\mathcal{A}}$, and so $\mathcal{A}$ is truth-singular.

Then, in case C is not $\sim$-subclassical, by Theorem 6.66, we have $C' = C$. Now, assume C is $\sim$-subclassical, in which case either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$, and so $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$, for, otherwise, we would have $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = 0$, in which case we would get $(\langle \frac{1}{2}, 1\rangle \veebar^{\mathfrak{A}^2} \sim^{\mathfrak{A}^2} \langle \frac{1}{2}, 1\rangle) = \langle 0,1\rangle \notin K_4 \supseteq K_3$, and so neither $K_3 \ni \langle \frac{1}{2}, 1\rangle$ nor K_4 would form a subalgebra of $\mathfrak{A}^2$. Further, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{E}$ of it, being a strict homomorphic counter-image of a strict homomorphic image of $\mathcal{D}$, and so a $(\veebar, \sim)$-paracomplete (in particular, consistent, in which case $I \neq \varnothing$) truth-non-empty model of C', in view of (2.23), for $\mathcal{D}$ is so. Hence, $C' \subseteq C$, by (2.23), Lemma 6.79(v)⇒(iv) and the following claim:

Claim 6.82. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and $\mathcal{E}$ a truth-non-empty $(\veebar, \sim)$-paracomplete subdirect product of it. Suppose both C is $\veebar$-disjunctive (viz., $\mathcal{A}$ is so) and either $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$ or $(I \times \{\frac{1}{2}\}) \in E$. Then, $\mathcal{A}$ is embeddable into $\mathcal{E}$, if $(I \times \{\frac{1}{2}\}) \in D$, and $\mathcal{K}'_3$ is embeddable into $\mathcal{E}$, otherwise.*

Proof. Then, by (2.23), $\mathcal{E} \in \mathrm{Mod}(C)$, in which case C is $(\veebar, \sim)$-paracomplete, for $\mathcal{E}$ is so, and so is $\mathcal{A}$. Therefore, $\mathcal{A}$, being $\veebar$-disjunctive with $1 \in D^{\mathcal{A}}$, is truth-singular, and so not $\sim$-paraconsistent, in which case, by Claim 6.39, E contains both $a \triangleq (I \times \{1\})$ and $b \triangleq (I \times \{0\})$. Consider the following complementary cases:

- $(I \times \{\frac{1}{2}\}) \in E$,
  in which case, as $I \neq \varnothing$, for $\mathcal{E}$, being $(\veebar, \sim)$-paracomplete, is consistent, $\{\langle e, I \times \{e\}\rangle \mid e \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$.
- $(I \times \{\frac{1}{2}\}) \notin E$,
  in which case $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$, and so $((1/0/\frac{1}{2}) \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}}(1/0/\frac{1}{2})) = (1/1/\frac{1}{2})$, for $\mathcal{A}$ is $\veebar$-disjunctive and $D^{\mathcal{A}} = \{1\}$. Hence, as $\mathcal{E}$ is $(\veebar, \sim)$-paracomplete, there is some $c \in E$ such that $d \triangleq (c \veebar^{\mathfrak{E}} \sim^{\mathfrak{E}} c) \notin D^{\mathcal{E}}$, in which case $d \in (E \cap \{\frac{1}{2}, 1\}^I) \subseteq E \not\ni (I \times \{\frac{1}{2}\})$, and so $I \neq J \triangleq \{i \in I \mid \pi_i(d) = \frac{1}{2}\} \neq \varnothing$. Given any $\bar{e} \in A^2$, set $(e_0 \| e_1) \triangleq ((J \times \{e_0\}) \cup ((I \setminus J) \times \{e_1\})) \in A^I$. In this way, $E \ni a = (1\|1)$, $E \ni b = (0\|0)$ and $E \ni d = (\frac{1}{2}\|1)$. Then, as $J \neq \varnothing \neq (I \setminus J)$ and $\{(x\|y) \mid \langle x, y\rangle \in K_3\} \subseteq E$, $\{\langle\langle x, y\rangle, (x\|y)\rangle \mid \langle x, y\rangle \in K_3'\}$ is an embedding of $\mathcal{K}_3'$ into $\mathcal{E}$. □

Thus, $C' = C$, as required. □

By Lemmas 6.79, 6.81, Corollaries 2.14, 3.21(ii) ⇒(i) and Remark 2.9(i)**d)**, we first get the following effective algebraic criterion of the maximal inferential $(\veebar, \sim)$-paracompleteness of $\veebar$-disjunctive $(\veebar, \sim)$-paracomplete Σ-logics with subclassical negation $\sim$ (cf. Corollary 6.29):

Theorem 6.83. *Suppose C is $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so; cf. Lemma 6.30). Then, C has no proper axiomatic/inferentially $(\veebar, \sim)$-paracomplete extension (i.e., C is maximally axiomatically/inferentially $(\veebar, \sim)$-paracomplete)./" iff either 2 does not form a subalgebra of $\mathfrak{A}$ or either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$."*

And what is more, we have the following effective algebraic criterion of their structural completeness:

Theorem 6.84. *Suppose C is $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so; cf. Lemma 6.30). Then, the following are equivalent:*

(i) *C is structurally complete;*
(ii) *C [has a theorem and] is maximally $(\veebar, \sim)$-paracomplete;*
(iii) *C has a theorem and, providing it is $\sim$-subclassical, either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$ (i.e., C {viz., $\mathcal{A}$} is not implicative; cf. Lemmas 6.31 and 6.79(i)⇔(iv));*
(iv) *both $\{\frac{1}{2}\}$ does not form a subalgebra of $\mathfrak{A}$ and either 2 does not form a subalgebra of $\mathfrak{A}$ or either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$.*

Proof. First, (i)⇒(iii) is by Remark 2.5 and Lemma 6.79(iv)⇒(v). Next, as $\mathcal{A}$ is then truth-singular, (iii)⇔(iv) is by Corollaries 3.21(i)⇔(iv), 6.51 and Remark 2.9(i)**d)**,(ii). Further, in case C has a theorem, any extension of it has a theorem, and so is $(\veebar, \sim)$-paracomplete iff it is inferentially so. Therefore, (iii)⇒(ii) is by Lemma 6.81. Finally, (ii)⇒(i) is by Lemma 6.78. □

Corollary 6.85. *Let $\Sigma \supseteq \Sigma_{+,\sim[,01]}$, $\mathcal{A}'$ a Σ-expansion of $\mathcal{DM}_{4[,01]}$ and C' the logic of $\mathcal{A}'$. Suppose C' has a theorem, while $\mathfrak{A}'$ is regular, whereas $DM_{3,1}$ forms a subalgebra of it (in particular, $\Sigma = \Sigma_{+,\sim,01}$). Then, $C'_{3,1}$ is the structural completion of C'.*

Proof. In that case, e_3 is an isomorphism from the canonical truth-singular three-valued $\sim$-super-classical Σ-matrix $\mathcal{A}$ with $\mathfrak{A} \triangleq e_3^{-1}[\mathfrak{A}'_{3,1}]$ onto $\mathcal{A}'_{3,1}$, in which case,

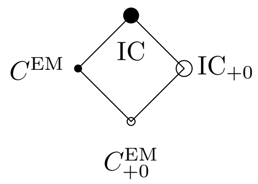

FIGURE 1. The lattice of proper extensions of C.

by (2.23), the logic C of $\mathcal{A}$ is equal to $C'_{3,1} \supseteq C'$, and so is both $\vee$-disjunctive (cf. Remark 2.9(ii)) and $(\vee, \sim)$-paracomplete as well as has a theorem. Let us prove, by contradiction, that $\langle 1, 0\rangle \notin K'_3$. For suppose $\langle 1, 0\rangle \in K'_3$, in which case there is some $\varphi \in \mathrm{Fm}^3_\Sigma$ such that $\varphi^{\mathfrak{A}}(0, \frac{1}{2}, 1) = 1$ and $\varphi^{\mathfrak{A}}(0, 0, 1) = 0$, and so, applying $e_3 \in \hom(\mathfrak{A}, \mathfrak{A}')$ to these equalities, we have $\varphi^{\mathfrak{A}'}(\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 1\rangle) = \langle 1, 1\rangle$ and $\varphi^{\mathfrak{A}'}(\langle 0, 0\rangle, \langle 0, 0\rangle, \langle 1, 1\rangle) = \langle 0, 0\rangle$. Hence, as $\langle 0, 1\rangle \sqsubseteq \langle 0, 0\rangle$, while $\mathfrak{A}'$ is regular, whereas $\sqsubseteq$ is reflexive, we get $\langle 1, 1\rangle \sqsubseteq \langle 0, 0\rangle$. This contradiction shows that $\langle 1, 0\rangle \notin K'_3$. Therefore, by Lemma 6.79(iv)$\Rightarrow$(ii) and Theorem 6.84(iii)$\Rightarrow$(i), $C'_{3,1} = C$ is structurally complete. Finally, by Lemma 4.24 of [25] with $\mathcal{A}'$ instead of $\mathcal{A}$ and $\mathcal{B} = \mathcal{A}'_{3,1}$, $C'_{3,1}(\varnothing) = C'(\varnothing)$, as required. $\square$

This subsumes Theorem 6.11, providing a more generic insight into it.

Lemma 6.86. *Suppose C is $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so; cf. Lemma 6.30). Then, C^{EM} is $\sim$-classical, whenever C is $\sim$-subclassical, in which case $C^{\mathrm{EM}} = C^{\mathrm{PC}}$, and inconsistent, otherwise.*

Proof. Then, by Remark 2.9(i)**d)**,(ii), C is not $\sim$-classical, while there is a non-$(\veebar, \sim)$-paracomplete submatrix of $\mathcal{A}$ iff 2 forms a subalgebra of $\mathfrak{A}$, in which case $\mathcal{A}{\upharpoonright}2$ is the only non-$(\veebar, \sim)$-paracomplete submatrix of $\mathcal{A}$. In this way, Corollaries 2.14 and 6.51 complete the argument. $\square$

Finally, by (2.21), Remarks 2.4, 2.6, 2.7, 2.9(i)**d)**,(ii), Lemmas 6.30, 6.79, 6.81, 6.86, Corollaries 3.21(i)$\Leftrightarrow$(iv), 6.51 and Theorem 6.64, we also get:

Theorem 6.87. *Suppose C is both $\veebar$-disjunctive, $(\veebar, \sim)$-paracomplete and [not] $\sim$-subclassical as well as has a/no theorem. Then, proper (arbitrary/"merely non-pseudo-axiomatic") extensions of C form the four-element diamond (resp., two-element chain) [resp., $(2(-1))$-element chain] depicted at Figure 1 (with merely solid circles) [(and) with solely big circles] iff either C is not $\sim$-subclassical or, otherwise, either K_3 or K_4 forms a subalgebra of $\mathfrak{A}^2$ {"that is"/"in particular", C is not implicative}, $\mathrm{IC}_{\langle/+0\rangle}$ $|C^{\mathrm{EM}}_{\langle/+0\rangle}$ being $\veebar$-disjunctive, relatively axiomatized by $(\langle x_0 \vdash\rangle(x_1|(x_1 \veebar \sim x_1))$ and defined by $(\varnothing|\{\mathcal{A}{\upharpoonright}2\})\langle\cup\{\mathcal{A}{\upharpoonright}\{\frac{1}{2}\}\}\rangle$, respectively.*

Perhaps, most representative subclassical instances of this discussion are three-valued expansions (by solely classical constants $\bot$ and $\top$ interpreted by 0 and 1, respectively, as non-purely-inferential ones with $K_{4[-1]}$ [not] forming a subalgebra of $\mathfrak{A}^2$) of Kleene' logic [7], {the implication-free fragment of} Gödel's one [4] — with $K_{3[+1]}$ [not] forming a subalgebra of $\mathfrak{A}^2$ — and Łukasiewicz' one [9] (as an implicative one; cf. Example 7 of [22]), having a unique proper non-pseudo-axiomatic $(\veebar, \sim)$-paracomplete extension (cf. [24]). In this way, these instances (apart from the last — non-structurally-complete — one, covered by the next subsubsection) show that, as opposed to $\sim$-paraconsistent three-valued Σ-logics with subclassical negation $\sim$, the structural completeness of $\veebar$-disjunctive $(\veebar, \sim)$-paracomplete ones, though equally implying (even, being equivalent to) their maximal $(\veebar, \sim)$-paracompleteness, does not subsume absence of their $\sim$-classical extensions.

6.2.1. *Extensions of implicative paracomplete logics with subclassical negation and lattice disjunction and conjunction.* A Σ-matrix/-logic is said to be *[/maximally] $\sqsupset$-implicatively $\sim$-paracomplete*, provided the rule:

$$\{\sim^i x_0 \sqsupset \sim^{1-i} x_0 \mid i \in 2\} \vdash x_0 \tag{6.16}$$

is not satisfied in it [/and it has no proper $\sqsupset$-implicatively $\sim$-paracomplete extension], in which case it is "truth-non-empty and"/inferentially consistent. (Clearly, any $\sqsupset$-implicative $\sim$-negative/-classical Σ-matrix/-logic is not $\sqsupset$-implicatively $\sim$-paracomplete/, in view of Lemma 6.31.) By C^{INPC} we denote the least $\sqsupset$-implicatively non-$\sim$-paracomplete extension of C, that is, the extension of C relatively axiomatized by (6.16).

Throughout this subsubsection, it is supposed that C is both $\sqsupset$-implicative, $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (viz., $\mathcal{A}$ is so; cf. Lemmas 6.30, 6.31), in which case $(\{\frac{1}{2}, \sim^{\mathfrak{A}} \frac{1}{2}\} \cap D^{\mathcal{A}}) = \varnothing$ (in particular, $\mathcal{A}$ is truth-singular), and so $(\frac{1}{2} \sqsupset^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = 1 = (\sim^{\mathfrak{A}} \frac{1}{2} \sqsupset^{\mathfrak{A}} \frac{1}{2})$. In particular, $\mathcal{A}$ is $\sqsupset$-implicatively $\sim$-paracomplete (and so is C), for (6.16) is not true in it under $[x_0/\frac{1}{2}]$. And what is more, we have:

Theorem 6.88. *C is maximally $\sqsupset$-implicatively $\sim$-paraconsistent.*

Proof. Let C' be an $\sqsupset$-implicatively $\sim$-paracomplete extension of C, in which case $x_1 \notin T \triangleq C'(\{\sim^i x_0 \sqsupset \sim^{1-i} x_0 \mid i \in 2\})$, while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of $C' \supseteq C$, and so is its finitely-generated $\sqsupset$-implicatively $\sim$-paracomplete submatrix $\mathcal{B} \triangleq \langle \mathfrak{Fm}^2_{\Sigma}, T \cap \mathrm{Fm}^2_{\Sigma}\rangle$, in view of (2.23). Hence, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathbf{H}^{-1}(\mathbf{H}(\mathcal{B}))$ of it, in which case, by (2.23), $\mathcal{D}$ is an $\sqsupset$-implicatively $\sim$-paracomplete (and so both consistent and truth-non-empty) model of $C' \supseteq C$, and so, if $\mathcal{D}$ was not $(\veebar, \sim)$-paracomplete, then it would be a consistent [truth-non-empty] model of C^{EM}, in which case its logic C'' would be a[n inferentially] consistent extension of C^{EM}, and so, by Lemmas 6.31, 6.86 and Theorem 6.64, C'' would be both $\sim$-classical and $\sqsupset$-implicative, contrary to the fact that any $\sqsupset$-implicative $\sim$-classical Σ-logic is not $\sqsupset$-implicatively $\sim$-paracomplete. Therefore, $\mathcal{D}$ is $(\veebar, \sim)$-paracomplete. And what is more, since it is $\sqsupset$-implicatively $\sim$-paracomplete, there must be some $a \in D$ such that $\{a \sqsupset^{\mathfrak{D}} \sim^{\mathfrak{D}} a, \sim^{\mathfrak{D}} a \sqsupset^{\mathfrak{D}} a\} \subseteq D^{\mathcal{D}}$, in which case $D \ni a = (I \times \{\frac{1}{2}\})$, and so, by Claim 6.82, $\mathcal{A}$ is embeddable into $\mathcal{D}$. Thus, by (2.23), $C' = C$, as required. $\square$

Lemma 6.89. *Let $\mathcal{B}$ be a three-valued $\sim$-super-classical Σ-matrix, I a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{B})^I$, $\mathcal{D}$ a subdirect product of it and $J \triangleq \{i \in I \mid \frac{1}{2} \in \pi_i[D]\}$. Suppose $\mathfrak{B}$ is a $(\barwedge, \veebar)$-lattice with $0 \leq^{\mathfrak{B}}_{\barwedge} 1$ and $\frac{1}{2}(\leq \mid \not\leq)^{\mathfrak{B}}_{\barwedge} \sim^{\mathfrak{B}} \frac{1}{2}$, while $\mathcal{A}$ is weakly conjunctive, whenever it is $\sim$-paraconsistent, whereas $\mathcal{D}$ is truth-non-empty, otherwise. Then, there is some $a \in (D \cap \{\frac{1}{2}, 0|1\}^I)$ including $J \times \{\frac{1}{2}\}$.*

Proof. Then, by Claim 6.39, for each $j \in 2$, $(I \times \{j\}) \in D$. Moreover, $\langle B, \leq^{\mathfrak{B}}_{\barwedge}\rangle$ is a chain, for $|B| = 3$, in which case $\frac{1}{2}(\leq \mid \geq)^{\mathfrak{B}}_{\barwedge} \sim^{\mathfrak{B}} \frac{1}{2}$, while $\frac{1}{2}(\leq / \geq)^{\mathfrak{B}}_{\barwedge}(0|1)$. By induction on the cardinality of any $K \subseteq J$, let us prove that there is some $a \in (D \cap \{\frac{1}{2}, 0|1\}^I)$ including $K \times \{\frac{1}{2}\}$. In case $K = \varnothing$, we have $j \triangleq (0|1) \in 2$, while $(K \times \{\frac{1}{2}\}) = \varnothing \subseteq (I \times \{j\}) \in (D \cap \{\frac{1}{2}, 0|1\}^I)$. Now, assume $K \neq \varnothing$. Take any $j \in K \subseteq J$, in which case $L \triangleq (K \setminus \{j\}) \subseteq J$, while $|L| < |K|$, and so, by induction hypothesis, there is some $b \in (D \cap \{\frac{1}{2}, 0|1\}^I)$ including $L \times \{\frac{1}{2}\}$. Moreover, since $\frac{1}{2} \in \pi_j[D]$, there is some $c \in D$ such that $\pi_j(c) = \frac{1}{2}$. Let $d \triangleq (c(\barwedge|\veebar)^{\mathfrak{D}} \sim^{\mathfrak{D}} c) \in D$ and $a \triangleq (b(\barwedge/\veebar)^{\mathfrak{D}} d) \in D$. Then, as $0 \leq^{\mathfrak{B}}_{\barwedge} 1$, while $\frac{1}{2}(\leq \mid \geq)^{\mathfrak{B}}_{\barwedge} \sim^{\mathfrak{B}} \frac{1}{2}$, for each $i \in I$, $\pi_i(d)$ is equal to $\frac{1}{2}$, if $\pi_i(c)$ is so, and is equal to $0|1$, otherwise, in which case, as $b \in \{\frac{1}{2}, 0|1\}^I$, while $\frac{1}{2}(\leq / \geq)^{\mathfrak{B}}_{\barwedge}(0|1)$, $\pi_i(a)$ is equal to $\frac{1}{2}$, if either $\pi_i(b)$ or $\pi_i(d)$ is so, and is equal to $0|1$, otherwise, and so $a \in (D \cap \{\frac{1}{2}, 0|1\}^I)$ includes $K \times \{\frac{1}{2}\}$, for $K = (L \cup \{j\})$. Thus, the case, when $K = J$, completes the argument. $\square$

Throughout the rest of this subsubsection, it is also supposed that $\mathfrak{A}$ is a $(\bar{\wedge}, \underline{\vee})$-lattice, in which case it is a chain one with unit 1, for $\mathcal{A}$ is three-valued, truth-singular and $\underline{\vee}$-disjunctive, and so $\mathcal{A}$ is $\bar{\wedge}$-conjunctive.

Corollary 6.90. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, and $\mathcal{D}$ an $\sqsupset$-implcatively non-$\sim$-paraconsistent consistent subdirect product of $\overline{\mathcal{C}}$. Then, 2 forms a subalgebra of $\mathfrak{A}$, while* $\hom(\mathcal{D}, \mathcal{A}{\upharpoonright}2) \neq \varnothing$.

Proof. In that case, by (2.12) and Corollary 3.21(iv)$\Rightarrow$(i), $\mathcal{D}$ is truth-non-empty. Therefore, if, for each $i \in I$, $\frac{1}{2}$ was in $\pi_i[D] = C_i$, then, by Lemma 6.89 with $J = I$, $a \triangleq (I \times \{\frac{1}{2}\})$ would be in D, in which case (6.16) would not be true in $\mathcal{D}$ under $[x_0/a]$, for $I \neq \varnothing$, because $\mathcal{D}$ is consistent, and so $\mathcal{D}$ would be $\sqsupset$-implicatively $\sim$-paracomplete. Hence, there is some $i \in I$ such that $\frac{1}{2} \notin B \triangleq \pi_i[D] = C_i \neq \varnothing$, in which case $B \subseteq 2$ forms a subalgebra of $\mathfrak{A}$, and so $B = 2$, while $(\pi_i{\upharpoonright}D) \in \hom(\mathcal{D}, \mathcal{A}{\upharpoonright}B)$. $\square$

Theorem 6.91. *The following are equivalent:*

(i) C^{INPC} *is consistent;*
(ii) C^{INPC} *is $\sim$-subclassical;*
(iii) C^{INPC} *is $(\underline{\vee}, \sim)$-paracomplete;*
(iv) *C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$; cf. Corollary 6.51), in which case C^{INPC} is defined by $\mathcal{K}_6 \triangleq (\mathcal{A} \times (\mathcal{A}{\upharpoonright}2))$, and so $C^{\mathrm{INPC}}(\varnothing) = C(\varnothing)$.*

Proof. First, (i/iii) is a particular case of (ii[i]/iv), respectively, for $\mathcal{A}$ is $(\underline{\vee}, \sim)$-paracomplete. Next, (iv)$\Rightarrow$(ii) is by the consistency of $\mathcal{K}_6$, (2.12) and Theorem 6.64. Further, (i)$\Rightarrow$(iv) is by Theorem 6.63.

Finally, assume (i,iv) hold. Then, by Theorem 2.13 with $\mathsf{M} \triangleq \{\mathcal{A}\}$ and $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathsf{M}))$, C^{INPC} is finitely-defined by the non-empty class S of all consistent members of $\mathsf{K} \cap \mathrm{Mod}(C')$. Consider any $\mathcal{D} \in \mathsf{S} \subseteq \mathrm{Mod}\,(6.16)$, in which case there are some finite set I and some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ such that $\mathcal{D}$ is a subdirect product of it, and so, by Corollary 6.90, $\hom(\mathcal{D}, \mathcal{A}{\upharpoonright}2) \neq \varnothing$. Take any $g \in \hom(\mathcal{D}, \mathcal{A}{\upharpoonright}2)$. Consider any $a \in (D \setminus D^{\mathcal{D}})$. Then, there is some $i \in I$ such that $\pi_i(a) \notin D^{\mathcal{A}}$, while $f \triangleq (\pi_i{\upharpoonright}D) \in \hom(\mathcal{D}, \mathcal{A})$, in which case $h \triangleq (f \times g) \in J \triangleq \hom(\mathcal{D}, \mathcal{K}_6)$, while $h(a) \notin D^{\mathcal{K}_6}$, and so $(\prod J) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{K}_6^J)$. Thus, by (2.23), C^{INPC} is finitely-defined by the finite $\mathcal{K}_6$, in which case it, being finitary, for (6.16) is so, while $\mathcal{A}$ is finite, is defined by $\mathcal{K}_6$, and so (2.24) and the fact that $(\pi_0{\upharpoonright}K_6) \in \hom^{\mathrm{S}}(\mathcal{K}_6, \mathcal{A})$ complete the argument. $\square$

Lemma 6.92. *Suppose C is $\sim$-subclassical. Then, $\mathcal{K}_6$ is generated by $K_1 \triangleq \{\langle \frac{1}{2}, 1\rangle\}$.*

Proof. Let $\mathcal{D}$ be the submatrix of $\mathcal{K}_6$ generated by K_1. Then, by (2.12) and the truth-singularity of $\mathcal{A}$, we have $D \ni a \triangleq (\langle \frac{1}{2}, 1\rangle \sqsupset^{\mathfrak{D}} \langle \frac{1}{2}, 1\rangle) = \langle 1, 1\rangle$, in which case $D \ni \sim^{\mathfrak{D}} a = \langle 0, 0\rangle$, and so $K_3 = (\Delta_2 \cup K_1) \subseteq D$. Hence, by (2.12), Lemma 6.79(i)$\Rightarrow$(ii) and Claim 6.80, $D \ni b \triangleq \langle 1, 0\rangle$, in which case $D \ni \sim^{\mathfrak{D}} b = \langle 0, 1\rangle$, while there is some $\phi \in \mathrm{Fm}^1_\Sigma$ such that $\phi^{\mathfrak{A}}(\frac{1}{2}) = 1$, whereas $\phi^{\mathfrak{A}}(1) = 0$, in which case we have $\varphi \triangleq (x_0 \,\bar{\wedge}\, \phi) \in \mathrm{Fm}^1_\Sigma$ such that $D \ni \varphi^{\mathfrak{D}}(\langle \frac{1}{2}, 1\rangle) = \langle \frac{1}{2}, 0\rangle$, for $\mathfrak{A}$ is a $(\bar{\wedge}, \underline{\vee})$-lattice with unit 1, and so $K_6 = (K_3 \cup M_2 \cup \{\langle \frac{1}{2}, 0\rangle\}) \subseteq D$, as required. $\square$

As $\sim^{\mathfrak{A}} 1 = 0$, by Lemma 6.92, we immediately have:

Corollary 6.93. *Suppose C is $\sim$-subclassical, while $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$. Then, $\mathcal{K}_6$ is generated by $\{\langle \frac{1}{2}, 0\rangle\}$.*

Lemma 6.94. *Let $\mathcal{K}'_5$ be the submatrix of $\mathcal{A}^2$ generated by $K_5 \triangleq (K_6 \setminus K_1)$. Suppose C is $\sim$-subclassical. Then, $\mathcal{K}'_5$ is a model of any $(\veebar, \sim)$-paracomplete extension of C^{INPC}. In particular, the structural completion of $C^{[\mathrm{INPC}]}$ is defined by $\mathcal{K}'_5$.*

Proof. Then, by Theorem 6.91, $C^{\mathrm{INPC}}(\varnothing) = C(\varnothing)$. while C^{INPC} is defined by $\mathcal{K}_6$. Let C' be any $(\veebar, \sim)$-paracomplete (in particular, having same theorems, for C is so) extension of C^{INPC}, in which case, by (2.12), $(2.18) \notin T \triangleq C'(\varnothing) \supseteq C(\varnothing) \ni$ (2.12), while, by the structurality of C', $\langle \mathfrak{Fm}^{\omega}_{\Sigma}, T\rangle$ is a model of C' (in particular, of C), and so is its truth-non-empty $(\veebar, \sim)$-paracomplete finitely-generated submatrix $\mathcal{D} \triangleq \langle \mathfrak{Fm}^{1}_{\Sigma}, T \cap \mathrm{Fm}^{1}_{\Sigma}\rangle$, in view of (2.23). Therefore, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{E}$ of it, being a strict homomorphic counter-image of a strict homomorphic image of $\mathcal{D}$, and so a $(\veebar, \sim)$-paracomplete (in particular, consistent, in which case $I \neq \varnothing$) truth-non-empty model of C' (in particular, of (6.16)), in view of (2.23), for $\mathcal{D}$ is so. Then, since $\mathcal{A}$, being truth-singular, for it is $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete, is not $\sim$-paraconsistent, by Claim 6.39, E contains both $a \triangleq (I \times \{1\})$ and $b \triangleq (I \times \{0\})$. Moreover, if E contained $c \triangleq (I \times \{\frac{1}{2}\})$, then (6.16) would not be true in $\mathcal{E}$ under $[x_0/c, x_1/b]$, for $I \neq \varnothing$. Consider the following complementary cases:

- $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$.
  Then, by Lemma 6.92, $\mathcal{K}_6$ is generated by $K_3 \supseteq K_1$. Hence, as $c \notin E$, by Lemma 6.82, $\mathcal{K}_6$ is embeddable into $\mathcal{E} \in \mathrm{Mod}(C')$, and so, by (2.23), a model of C', and so is its submatrix $\mathcal{K}'_5$.
- $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) \neq \frac{1}{2}$,
  in which case $\sim^{\mathfrak{A}} \frac{1}{2} = 0$, and so $\flat^{\mathfrak{A}}_{\barwedge} = \frac{1}{2}$. Moreover, $\mathfrak{A}{\restriction}2$ is a $(\barwedge, \veebar)$-lattice with zero 0, for $\mathfrak{A}$ is that with unit 1. And what is more, since $\mathcal{E}$ is $(\veebar, \sim)$-paracomplete, $J \triangleq \{i \in I \mid \frac{1}{2} \in \pi_i[E]\} \neq \varnothing$. Given any $x, y \in A$, set $(x\|y) \triangleq ((J \times \{x\}) \cup ((I \setminus J) \times \{y\})) \in A^I$. Then, $E \ni (a/b) = (1/0\|1/0)$. Moreover, by Lemma 2.3, $\mathfrak{E}$, being finite, is a $(\barwedge, \veebar)$-lattice with zero $d \triangleq (\frac{1}{2}\|0) \in E$. Hence, $I \neq J$, for $c \notin E$. Then, $E \ni [\sim^{\mathfrak{E}}]\sim^{\mathfrak{E}} d = ([1-]0\|[1-]1)$. Thus, $\{(x\|y) \mid \langle x, y\rangle \in K_5\} \subseteq E$. In this way, since $J \neq \varnothing \neq (I \setminus J)$, $\{\langle\langle x, y\rangle, (x\|y)\rangle \mid \langle x, y\rangle \in D\}$ is an embedding of $\mathcal{K}'_5$ into $\mathcal{E} \in \mathrm{Mod}(C')$, in which case, by (2.23), $\mathcal{K}'_5 \in \mathrm{Mod}(C')$.

Moreover, as $K_5 \subseteq K_6$, while $\pi_0[K_5] = A$, $\mathcal{K}'_5$ is a submatrix of $\mathcal{K}_6$, while $(\pi_0{\restriction}K'_5) \in \mathrm{hom}^{\mathrm{S}}(\mathcal{K}'_5, \mathcal{A})$, in which case, by (2.23) and (2.24), $\mathcal{K}'_5$ is a model of $C^{[\mathrm{INPC}]}$ such that $\mathrm{Cn}^{\omega}_{\mathcal{K}'_5}(\varnothing) = C^{[\mathrm{INPC}]}(\varnothing)$, and so the structural completion of $C^{[\mathrm{INPC}]}$ is defined by it. □

Lemma 6.95. *Suppose 2 forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\sim$-subclassical; cf. Corollary 6.51). Then, (i)⇔(ii)⇔(iii)⇐(iv)⇔(v)⇔(vi), where:*

(i) *$((2.18)[x_0/(2.18)]) \in C(\varnothing)$;*
(ii) *neither $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$ nor $0 \leq^{\mathfrak{A}}_{\barwedge} \frac{1}{2}$;*
(iii) *$(\frac{1}{2} (\veebar)^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) \neq \frac{1}{2}$;*
(iv) *K_5 forms a subalgebra of $\mathfrak{A}^2$;*
(v) *$\mathfrak{K}_6$ is not generated by $K_2 \triangleq \{\langle \frac{1}{2}, 0\rangle, \langle 1, 1\rangle\}$;*
(vi) *neither $\mathfrak{K}_6$ is generated by K_2 nor $\mathfrak{A}$ has a discriminator.*

In particular, K_5 does not form a subalgebra of $\mathfrak{A}^2$, whenever $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$.

Proof. First, (i)⇔(ii)⇔(iii) are immediate. Next, if $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$, then $(\langle \frac{1}{2}, 0\rangle \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \langle \frac{1}{2}, 0\rangle) = \langle \frac{1}{2}, 1\rangle \notin K_5$, in which case $K_5 \ni \langle \frac{1}{2}, 0\rangle$ does not form a subalgebra of $\mathfrak{A}^2$, and so (iv)⇒(iii) holds.

Further, assume K_5 does not form a subalgebra of A^2, in which case $K'_5 = K_6$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}^2$ generated by $K_2 \subseteq K_6$. Then, in case $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$, by Corollary 6.93, $\mathfrak{K}_6$ is generated by $K_2 \ni \langle \frac{1}{2}, 0 \rangle$. Otherwise, $\sim^{\mathfrak{A}} \frac{1}{2} = 0$, in which case $B \ni \sim^{\mathfrak{B}} \langle \frac{1}{2}, 0 \rangle = \langle 0, 1 \rangle$, and so $K_5 = (K_2 \cup \Delta_2 \cup M_2) \subseteq B$, in which case $K_6 = K'_5 \subseteq B \subseteq K_6$, and so $B = K_6$. Thus, (v)⇒(iv) holds. Furthermore, (v) is a particular case of (vi). Finally, if $\mathfrak{K}_6$ is generated by $K_2 \subseteq K_5$, then $K_6 \subseteq K'_5 \subseteq K_6$, in which case $K'_5 = K_6$, and so K_5 does not form a subalgebra of $\mathfrak{A}^2$, for, otherwise, $K'_5 \supseteq K_5$ would be equal to $K_5 \neq K_6$. Likewise, if $\mathfrak{A}$ has a discriminator δ, then so does $\mathfrak{A} \upharpoonright 2$, in which case δ is a congruence-permutation term for both $\mathfrak{A}$ and $\mathfrak{A} \upharpoonright 2$, being simple, and so so K_5 does not form a subalgebra of $\mathfrak{A}^2$, for, otherwise, $\mathfrak{D} \triangleq (\mathfrak{A}^2 \upharpoonright K_5)$ would be a subdirect product of $\langle \mathfrak{A}, \mathfrak{A} \upharpoonright 2 \rangle$, in which case, by Lemma 2.2, it would be isomorphic to either $\mathfrak{K}_6$ or $\mathfrak{A}$ or $\mathfrak{A} \upharpoonright 2$, and so $5 = |D|$ would be equal to either $6 = |K_6|$ or $3 = |A|$ or $2 = |2|$. Thus, (iv)⇒(vi) holds, as required. □

Next, by $C^{\mathrm{INPC+DN}}$ we denote the extension of C^{INPC} relatively axiomatized by the *Double Negation* rule:

$$\sim\sim x_0 \vdash x_0, \tag{6.17}$$

the inverse one being satisfied in C.

Lemma 6.96. *Suppose C is $\sim$-subclassical* ⟨*i.e., 2 forms a subalgebra of $\mathfrak{A}$; cf. Corollary 6.51*⟩ *(while $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$ {in particular, K_5 forms a subalgebra of $\mathfrak{A}^2$; cf. Lemma 6.95}). Then,* (6.17) *is ([not]) true in $\mathcal{A} \upharpoonright 2$ ([resp. $\mathcal{K}_6$] {as well as in $\mathcal{K}_5 \triangleq (\mathcal{A}^2 \upharpoonright K_5)$}).*

Proof. First, (6.17) is true in $\mathcal{A} \upharpoonright 2$, for $\sim^{\mathfrak{A}} \sim^{\mathfrak{A}} i = i$, for all $i \in 2$. Finally, using the truth-singularity of $\mathcal{A}$, it is routine checking that (6.17) is ([not]) true in $\mathcal{A}^2$ under $[x_0 / \langle \frac{1}{2}, [1-]0 \rangle]$. □

Theorem 6.97. *Suppose C is $\sim$-subclassical. Then, the following are equivalent:*

(i) *C^{INPC} has a proper $(\veebar, \sim)$-paracomplete extension;*
(ii) *C^{INPC} is not structurally complete;*
(iii) *$C^{\mathrm{INPC}} \neq C^{\mathrm{INPC+DN}} \neq C^{\mathrm{PC}}$;*
(iv) *$C^{\mathrm{INPC+DN}} \neq C^{\mathrm{INPC}}$ is $(\veebar, \sim)$-paracomplete;*
(v) *K_5 forms a subalgebra of $\mathfrak{A}^2$,*

in which case $\frac{1}{2} \leq^{\mathfrak{A}}_{\wedge} 0 = \sim^{\mathfrak{A}} \frac{1}{2}$, while the logic of $\mathcal{K}_5$ has no proper $(\veebar, \sim)$-paracomplete extension, whereas it is the structural completion of $C^{[\mathrm{INPC}]}$.

Proof. First, since $(K_6 \setminus K_5) = K_1$ is a singleton, K_5 forms a subalgebra of $\mathfrak{A}^2$ iff $K'_5 \neq K_6$. In this way, (2.12), (2.23), Remark 2.9(i)**d)**,(ii), Corollaries 6.51, Lemmas 6.86, 6.94, 6.95, 6.96 and Theorem 6.91 complete the argument. □

Given any $\varphi \in \mathrm{Fm}^1_\Sigma$, by $C^{\mathrm{INPC}+\varphi}$ we denote the extension of C^{INPC} relatively axiomatized by:

$$\varphi \vdash x_0. \tag{6.18}$$

In this way, $C^{\mathrm{INPC+DN}} = C^{\mathrm{INPC}+(\sim\sim x_0)}$. A *characteristic formula for* a $K \subseteq (K_6 \setminus \Delta_2)$ is any $\varphi \in \mathrm{Fm}^1_\Sigma$ such that, for all $a \in K_6$, it holds that $(a \in K) \Rightarrow (\varphi^{\mathfrak{K}_6}(a) = \langle 1, 1 \rangle) \Rightarrow (a \neq \langle 0, 0 \rangle)$, in which case, unless $K = \varnothing$, (6.18) is not true in $\mathcal{K}_6$ under $[x_0/a]$, where $a \in K \not\ni \langle 1, 1 \rangle$.

Lemma 6.98. *Let φ be any characteristic formula for K_1 (in particular, $\varphi = \sim\sim x_0$, unless $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$). Then, C^{INPC} has no proper extension not satisfying* (6.18) *(in particular,* (6.17)*, unless $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$). In particular, $C^{\mathrm{INPC}+\varphi} = C^{\mathrm{INPC+DN}}$, unless $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$.*

Proof. The case, when C^{INPC} is inconsistent, is evident. Now, assume it is consistent. Then, by Theorem 6.91, C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$; cf. Corollary 6.51), while C^{INPC} is defined by $\mathcal{K}_6$. Consider any extension C' of

C^{INPC} not satisfying (6.18). Then, by Theorem 2.13, there are some set I, some $\overline{\mathcal{C}} \in \mathbf{S}(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of it, not satisfying (6.18), for this is finitary, in which case there is some $a \in (D \setminus D^{\mathcal{D}})$ such that $\varphi^{\mathfrak{D}}(a) \in D^{\mathcal{D}}$, and so $(I \times \{1\}) \neq a \in \{\frac{1}{2}, 1\}^I$, for $\varphi^{\mathfrak{A}}(0) \neq 1$. Hence, $\varnothing \neq J \triangleq \{i \in I \mid \pi_i(a) = \frac{1}{2}\} \neq I$, for, otherwise, (6.16) would not be true in $\mathcal{D}$ under $[x_i/a]_{i \in 2}$. Given any $\bar{a} \in A^2$, set $(a_0 \| a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. Then, $D \ni a \triangleq (\frac{1}{2} \| 1)$. In this way, as $J \neq \varnothing \neq (I \setminus J)$, by Lemma 6.92, $\{\langle \bar{a}, (a_0 \| a_1) \rangle \mid \bar{a} \in K_6\}$ is an embedding of $\mathcal{K}_6$ into $\mathcal{D} \in \mathrm{Mod}(C')$, in which case, by (2.23), $\mathcal{K}_6 \in \mathrm{Mod}(C')$, and so $C' = C^{\mathrm{INPC}}$. Finally, the fact that (6.18) is not true in $\mathcal{K}_6$ under $[x_0/\langle \frac{1}{2}, 1 \rangle]$ completes the argument. □

Finally, combining (2.12), Theorems 6.64, 6.66, 6.88, 6.91, 6.97, Lemmas 6.86, 6.95, 6.98 and Corollary 6.51, we get:

Theorem 6.99. *Suppose C is [not] non-$\sim$-subclassical [i.e., 2 forms a subalgebra of $\mathfrak{A}$, while K_5 is (not) non-$\mathfrak{A}^2$-closed (in which case $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$, whereas $C^{\mathrm{INPC+DN}}$ is {not} defined by $\mathcal{K}_5$)]. Then, the following hold:*

(i) *[({some of})] extensions of C form the (2[+2(+1{+1})])-element chain $C \subsetneq C^{\mathrm{INPC}} = [\mathrm{Cn}^{\omega}_{\mathcal{K}_6} \subsetneq (C^{\mathrm{INPC+DN}} = \{\subsetneq\}\, \mathrm{Cn}^{\omega}_{\mathcal{K}_5} \subsetneq)] C^{\mathrm{EM}} = [C^{\mathrm{PC}} = \mathrm{Cn}^{\omega}_{\mathcal{A}\restriction 2} \subsetneq$] IC [({others being simultaneously extensions of $C^{\mathrm{INPC+DN}}$ and sublogics of $\mathrm{Cn}^{\omega}_{\mathcal{K}_5}$})];*

(ii) *C is [(not) pre]maximally $(\veebar, \sim)$-paracomplete;*

(iii) *$C^{[INPC]}[(\cup\, \mathrm{Cn}^{\omega}_{\mathcal{K}_5})]$ is the structural completion of C.*

The []-optional ()-non-optional particular case of Theorem 6.99, covering the both $\sim$-subclassical and implicative (cf. Example 7 of [22]) Ł$_3$ [9], equally ensues from Theorem 3.3 of [20], Corollaries 4.6, 4.12 and 4.13 with $\Lambda = \{\barwedge, \veebar\}$ of [24], the fact that the poset $\langle A, \leq^{\mathfrak{A}}_{\barwedge} \rangle$ is a chain, while $\mathfrak{A}$ is generated by $\{\frac{1}{2}, 1\}$, for $0 = \sim^{\mathfrak{A}} 1$, and is a $(\barwedge, \veebar)$-lattice with unit 1, Remark 5.13(v), Lemma 6.95(v)⇒(iv) and the following observation:

Remark 6.100. [Suppose $0 \leq^{\mathfrak{A}}_{\barwedge} \frac{1}{2}$, while 2 forms a subalgebra of $\mathfrak{A}$ (i.e., C is $\sim$-subclassical; cf. Corollary 6.51). Then, $\mathfrak{A}$ is is a $(\barwedge, \veebar)$-lattice with unit 1 and zero 0. Moreover,] $\Upsilon \triangleq \{x_0, \sim x_0\}$ is a unary unitary equality determinant for $\mathcal{A}$, because $\sim^{\mathfrak{A}} \frac{1}{2} \notin D^{\mathcal{A}} = \{1\} \not\ni 0$, while $\sim^{\mathfrak{A}} i = (1 - i)$, for all $i \in 2$, in which case, by Remark 5.13(iv), $\{\phi \sqsupset \psi \mid (\phi \vdash \psi) \in \varepsilon_{\Upsilon}\}$ is an axiomatic binary equality determinant for $\mathcal{A}$, and so is $(x_0 \leftrightarrow x_1) \triangleq (\bar{\wedge} \langle \bar{\wedge} \langle \sim^i x_j \sqsupset \sim^i x_{1-j} \rangle_{j \in 2} \rangle_{i \in 2})$, in view of the $\barwedge$-conjunctivity of $\mathcal{A}$. Therefore, since $\mathcal{A}$ is $\sqsupset$-implicative, by Remark 5.13(v), $(x_0 \approx (x_0 \sqsupset x_0))[x_0/((x_0 \leftrightarrow x_1) \sqsupset (x_2 \leftrightarrow x_3))]$ is an implicative system for $\mathfrak{A}$. [And what is more, $\mathcal{A}$, being $\sqsupset$-implicative and truth-singular, is $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \sqsupset \sim(x_0 \sqsupset x_0))$, while $(\mathrm{img}\, \neg^{\mathfrak{A}}) \subseteq 2$, for 2 forms a subalgebra of $\mathfrak{A}$, in which case $(\sim^{\mathfrak{A}} \circ \neg^{\mathfrak{A}}) = \chi^{\mathcal{A}}$, and so $((\sim\neg(x_0 \leftrightarrow x_1) \barwedge x_2) \veebar (\neg(x_0 \leftrightarrow x_1) \barwedge x_0))$ is a discriminator for $\mathfrak{A}$.] □

In this connection,] recall that it is this alternative argumentation (more specifically, its "discriminator" particular case based upon Corollary 4.12 of [24]) that has been invoked therein to find the lattice of extensions of Ł$_3$ upon the basis of Example B.2 therein. On the other hand, the "discriminator" subcase does not at all exhaust the []-optional ()-non-optional case of Theorem 6.99, in view of the following double counterexample equally showing the possibility of the []-optional ()-optional case of this theorem:

Example 6.101. Let $\Sigma \triangleq (\Sigma_{+,\sim} \cup \{\top\})$, while $\mathcal{A}$ truth-singular with $\sim^{\mathfrak{A}} \frac{1}{2} \triangleq (0[+\frac{1}{2}])$, $\top^{\mathfrak{A}} \triangleq 1$, $\barwedge \triangleq \wedge$, $\veebar \triangleq \vee$ and $\frac{1}{2} \leq^{\mathfrak{A}}_{\barwedge} 0 \leq^{\mathfrak{A}}_{\barwedge} 1$ [whereas $\mathfrak{B}$ the Σ-algebra with $(\mathfrak{B} \restriction (\Sigma \setminus \{\sim\})) \triangleq (\mathfrak{D}_{2,01} \restriction (\Sigma \setminus \{\sim\}))$ and $\sim^{\mathfrak{B}} \triangleq \Delta_2$]. Then, 2 forms a subalgebra of $\mathfrak{A}$, in which case 2^2 forms a subalgebra of $\mathfrak{A}^2$, and so $K_5 = (2^2 \cup \{\langle \frac{1}{2}, 0\rangle\})$, [though] forming a subalgebra of $(\mathfrak{A}\restriction(\Sigma \setminus \{\sim\}))^2$, does [not] form a subalgebra of $\mathfrak{A}^2$, for $\langle 0[+\frac{1}{2}], 1\rangle = \sim^{\mathfrak{A}^2}\langle \frac{1}{2}, 0\rangle$ does [not] belong to K_5, while, by Theorem 6.50, C is $\sim$-subclassical, whereas, by Lemma 6.79(ii)⇒(i), $\mathcal{A}$ is implicative, for $\top \in C(\varnothing)$, while $\langle 1, 0\rangle = \sim^{\mathfrak{A}^2}(\langle \frac{1}{2}, 1\rangle \veebar^{\mathfrak{A}^2} \sim^{\mathfrak{A}^2}\top^{\mathfrak{A}^2}) \in K_3' \supseteq K_3 \supseteq K_1$. [And what is more, $\chi_A^2 \in \hom(\mathfrak{A}, \mathfrak{B})$ is surjective. Therefore, if $\mathfrak{A}$ had a discriminator, then this would be a congruence-permutation term for $\mathfrak{B}$, being simple, for it is two-element, in which case, by Lemma 2.2, the subdirect square $\mathfrak{D} \triangleq (\mathfrak{B}^2\restriction(2^2 \setminus \{\langle 0, 1\rangle\}))$ of $\mathfrak{B}$ would be isomorphic to either $\mathfrak{B}$ or $\mathfrak{B}^2$, and so $3 = |D|$ would be even.] Thus, anyway, $\mathfrak{A}$ has no discriminator, in view of Lemma 6.95(iv)⇒(vi). □

This — in addition to Subsection 5.5 of [24] — highlights the "non-discriminator" advance of the mentioned study.

In this way, Remarks 2.5, 2.7, Corollary 6.29, Lemmas 6.30, 6.79 and Theorems 6.84 and 6.99 exhaust the issue of structural completions of $\veebar$-disjunctive $(\veebar, \sim)$-paracomplete Σ-logics with subclassical negation $\sim$.

6.2.2. *Three-valued paraconsistent logics with subclassical negation and lattice conjunction and disjunction.* Throughout this subsubsection, it is supposed that:

- $\mathfrak{A}$ is a $(\barwedge, \veebar)$-lattice, in which case $\langle A, \leq^{\mathfrak{A}}\rangle$ is a chain poset for $|A| = 3$, and so $\mathfrak{A}$, being finite, is a distributive $(\barwedge, \veebar)$-lattice with zero and unit;
- $\mathcal{A}$ is $\sim$-paraconsistent (and so false-singular) and $\barwedge$-conjunctive, in which case $\flat^{\mathfrak{A}}_{\barwedge} = 0$, and so $\mathcal{A}$ is $\veebar$-disjunctive (in particular, C is maximally $\sim$-paraconsistent [cf. Corollary 6.47], while it is $\sim$-subclassical iff 2 forms a subalgebra of $\mathfrak{A}$, in which case C^{PC} is defined by $\mathcal{A}\restriction 2$ [cf. Corollary 6.51]);
- *unless otherwise specified*, $\sqsupset$ is the *material implication* $\sqsupset^{\sim}_{\veebar}$, in which case, by (2.8) satisfied in C, in view of its $\veebar$-disjunctivity, we have $C^{\mathrm{NP}} \subseteq C^{\mathrm{MP}}$, and so C, being $\sim$-paraconsistent, is not (weakly) $\sqsupset$-implicative.

Lemma 6.102. *Let I be a finite set, $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and $\mathcal{B}$ a consistent non-$\sim$-paraconsistent subdirect product of $\overline{\mathcal{C}}$. Then, 2 forms a subalgebra of $\mathfrak{A}$ and* $\hom(\mathcal{B}, \mathcal{A}\restriction 2) \neq \varnothing$.

Proof. First, by Lemma 6.89 with $J = I$, if $\frac{1}{2}$ was in $\pi_i[D] = C_i$, for each $i \in I$, then $a \triangleq (I \times \{\frac{1}{2}\})$ would be in D, in which case (2.16) would not be true in $\mathcal{D}$ under $[x_0/a, x_1/b]$, where $b \in (D \setminus D^{\mathcal{D}}) \neq \varnothing$, for $\mathcal{D}$ is consistent, and so $\mathcal{D}$ would be $\sim$-paraconsistent. Hence, there is some $i \in I$ such that $\frac{1}{2} \notin B \triangleq \pi_i[D] = C_i \neq \varnothing$, in which case $B \subseteq 2$ forms a subalgebra of $\mathfrak{A}$, and so $B = 2$, while $(\pi_i\restriction D) \in \hom(\mathcal{D}, \mathcal{A}\restriction B)$. □

Theorem 6.103. *Suppose C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$; cf. Corollary 6.51). Then, C^{NP} is defined by $\mathcal{L}_6 \triangleq (\mathcal{A} \times (\mathcal{A}\restriction 2))$, in which case $C^{\mathrm{NP}}(\varnothing) = C(\varnothing)$.*

Proof. Then, by Theorems 6.63 and 2.13 with $\mathsf{M} \triangleq \{\mathcal{A}\}$ and $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}_{\omega}(\mathbf{S}_*(\mathsf{M}))$, C^{NP} is finitely-defined by the non-empty class S of all consistent members of $\mathsf{K} \cap \mathrm{Mod}(C^{\mathrm{NP}})$. Consider any $\mathcal{D} \in \mathsf{S} \subseteq \mathrm{Mod}\,(2.16)$, in which case there are some finite set I and some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ such that $\mathcal{D}$ is a subdirect product of it, and so, by Lemma 6.102, $\hom(\mathcal{D}, \mathcal{A}\restriction 2) \neq \varnothing$. Take any $g \in \hom(\mathcal{D}, \mathcal{A}\restriction 2)$. Consider any $a \in (D \setminus D^{\mathcal{D}})$. Then, there is some $i \in I$ such that $\pi_i(a) \notin D^{\mathcal{A}}$, while $f \triangleq (\pi_i\restriction D) \in \hom(\mathcal{D}, \mathcal{A})$, in which case $h \triangleq (f \times g) \in J \triangleq \hom(\mathcal{D}, \mathcal{L}_6)$, while $h(a) \notin D^{\mathcal{L}_6}$, and so $(\prod J) \in \hom_{\mathrm{S}}(\mathcal{D}, \mathcal{L}_6^J)$. Thus, by (2.23), C^{NP} is finitely-defined by the finite $\mathcal{L}_6$, in which case it, being finitary, for (2.16) is so, while $\mathcal{A}$ is finite, is defined by $\mathcal{L}_6$, and so (2.24) and the fact that $(\pi_0\restriction L_6) \in \hom^{\mathrm{S}}(\mathcal{L}_6, \mathcal{A})$ complete the argument. □

Theorem 6.104. *C^{MP} is consistent iff C is $\sim$-subclassical, in which case $C^{\mathrm{NP}} \subsetneq C^{\mathrm{MP}} = C^{\mathrm{PC}}$, and so C^{NP} is not $\veebar$-disjunctive.*

Proof. First, if C^{MP} is consistent, then so is its sublogic C^{NP} (in view of (2.8) satisfied in C), in which case C is $\sim$-subclassical, by Theorem 6.63. Conversely, assume C is $\sim$-subclassical, in which case, by Corollary 6.51, 2 forms a subalgebra of $\mathfrak{A}$, while C^{PC} is defined by $\mathcal{A}{\restriction}2$. Then, by Remark 2.9(i)**c)**,(ii), $\mathcal{A}{\restriction}2$ is $\sqsupset^{\sim}_{\veebar}$-implicative, and so is C^{PC}, in which case $C^{\mathrm{MP}} \subseteq C^{\mathrm{PC}}$. For proving the converse, consider the following complementary cases:

- $C^{\mathrm{PC}}(\varnothing) = C(\varnothing)$.
  Then, Lemma 3.28 yields the fact that $C^{\mathrm{PC}} \subseteq C^{\mathrm{MP}}$.
- $C^{\mathrm{PC}}(\varnothing) \neq C(\varnothing)$.

  1st argument. Then, by Lemma 6.69**b)**$\Rightarrow$**e)**, $\mathcal{A}$ is implicative. First, we prove, by contradiction, that there is some $\varphi \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing))$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$. For suppose, for all $\varphi \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing))$, $\varphi^{\mathfrak{A}}(\frac{1}{2}) \neq \frac{1}{2}$, in which case $\varphi^{\mathfrak{A}}(\frac{1}{2}) \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$, and so $\varphi^{\mathfrak{A}}(\frac{1}{2}) = 1$. In particular, since $\mathcal{A}$ is both $\veebar$-disjunctive and, being false-singular, weakly $\sim$-negative, it is not $(\veebar, \sim)$-paracomplete, in view of Remark 2.9(i)**d)**, in which case $(\frac{1}{2} \veebar^{\mathfrak{A}} \sim^{\mathfrak{A}} \frac{1}{2}) = 1$, and so
$$\tfrac{1}{2} \leq^{\mathfrak{A}}_{\barwedge} 1 = \sim^{\mathfrak{A}} \tfrac{1}{2}, \tag{6.19}$$
  in view of the linearity of the poset $\langle A, \leq^{\mathfrak{A}}_{\barwedge} \rangle$. Consider any $\phi \in C(\varnothing)$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$. Let $U_a \triangleq (V_\omega \cap h^{-1}[\{a\}])$, where $a \in A$, and σ the Σ-substitution extending $(U_{\frac{1}{2}} \times \{x_0\}) \cup (U_1 \times \{\sim x_0\}) \cup (U_0 \times \{\sim\sim x_0\})$, in which case, by the structurality of C, we have $\psi \triangleq \sigma(\phi) \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing))$, and so, by (6.19), we get $h(\phi) = \psi^{\mathfrak{A}}(\frac{1}{2}) = 1$. Hence, $\mathcal{B} \triangleq \langle \mathfrak{A}, \{1\} \rangle \in \mathrm{Mod}_1(C)$. Let $\supset$ be any (possibly, secondary) binary connective of Σ, such that $\mathcal{A}$ is $\supset$-implicative, and $(x_0 \sqsupset x_1) \triangleq ((x_0 \supset x_1) \barwedge (x_0 \sqsupset^{\sim} x_1))$, in which case $\mathcal{A}$ is $\sqsupset$-implicative, for it is $\supset$-implicative, $\barwedge$-conjunctive, $\veebar$-disjunctive and false-singular, and so $(1 \sqsupset^{\mathfrak{A}} 0) = 0$. Moreover, $(1 \supset^{\mathfrak{A}} \frac{1}{2}) \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$, in which case, by (6.19), we have $\frac{1}{2} \leq^{\mathfrak{A}}_{\barwedge} (1 \supset^{\mathfrak{A}} \frac{1}{2})$, and so we get $(1 \sqsupset^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$, for $\sim^{\mathfrak{A}} 1 = 0 \leq^{\mathfrak{A}}_{\barwedge} \frac{1}{2}$. Therefore, (2.11) is true in $\mathcal{B} \in \mathrm{Mod}_1(C)$, in which case, by Lemma 3.28, $\mathcal{B} \in \mathrm{Mod}(C)$ is both finite and, by (6.19), $\veebar$-disjunctive, and so, by Remarks 2.8(ii), 2.9(i)**d)** and Corollaries 3.20 and 6.36, there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$. Then, $h(0) = h(\frac{1}{2}) = 0$, in which case $0 = h(0) = h(\frac{1}{2} \supset^{\mathfrak{A}} 0) = (h(\frac{1}{2}) \supset^{\mathfrak{A}} h(0)) = (0 \supset^{\mathfrak{A}} 0) \in D^{\mathcal{A}}$, and so this contradiction shows that there is some $\varphi \in (\mathrm{Fm}^1_\Sigma \cap C(\varnothing)) \subseteq C^{\mathrm{MP}}(\varnothing)$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2}) = \frac{1}{2}$. Hence, $\sim^{\mathfrak{A}} \varphi^{\mathfrak{A}}(\frac{1}{2}) = \sim^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}}$, for $\mathcal{A}$ is $\sim$-paraconsistent, in which case $\sim\varphi \veebar (2.17)$ is true in $\mathcal{A}$ under any $[x_0/\frac{1}{2}, x_1/a]$, where $a \in A$, for $\mathcal{A}$ is $\veebar$-disjunctive, and so, since (2.17) is true in $\mathcal{A}$ under any $[x_0/i, x_1/a]$, where $i \in 2$, $\sim\varphi \veebar (2.17)$ is true in $\mathcal{A}$. Thus, $(\sim\varphi \veebar (2.17)) \in C(\varnothing) \subseteq C^{\mathrm{MP}}(\varnothing)$, in which case, by the structurality of C^{MP} and $(2.11)[x_0/\varphi, x_1/(2.17)]$, (2.17) is satisfied in C^{MP}, and so, by Corollary 6.53, $C^{\mathrm{PC}} \subseteq C^{\mathrm{MP}}$.

  2nd argument. Then, by Lemma 6.69**b)**$\Rightarrow$**c)**, $\langle 0, 1 \rangle \in K'_{3,i}$, for each $i \in 2$, in which case there is some $\varphi_i \in \mathrm{Fm}^3_\Sigma$ such that $\varphi_i^{\mathfrak{A}}(0, \frac{1}{2}[-\frac{1}{2} + i], 1) = (0[+1])$. Moreover, by Theorem 2.13 with $\mathsf{M} \triangleq \{\mathcal{A}\}$ and $\mathsf{K} \triangleq \mathbf{P}^{\mathrm{SD}}_\omega(\mathbf{S}_*(\mathsf{M}))$, C^{MP} is finitely-defined by $\mathsf{S} \triangleq (\mathsf{K} \cap \mathrm{Mod}(C^{\mathrm{MP}}))$. Consider any $\mathcal{D} \in \mathsf{S} \subseteq \mathrm{Mod}\,(2.11)$, in which case there are some finite set I and some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ such that $\mathcal{D}$ is a subdirect product of it. Let $J \triangleq \{i \in I \mid \frac{1}{2} \in \pi_i[D]\}$. Given any $\bar{a} \in A^2$, set $(a_0 \| a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. Then, by Claim 6.39, $D \ni (a/b) \triangleq (0/1 \| 0/1)$. Moreover, by Lemma 6.89, $D \ni c \triangleq (\frac{1}{2} \| 0|1)$, whenever $\frac{1}{2}(\leq \mid \not\leq)^{\mathfrak{A}}_{\wedge} \sim^{\mathfrak{A}} \frac{1}{2}$. Then, $D \ni d \triangleq \varphi^{\mathfrak{D}}_{0|1}(a,c,b) = (0\|1)$, in which case $D \ni e \triangleq (c \veebar^{\mathfrak{D}} d) = (\frac{1}{2}\|1)$, and so $(\sim^{\mathfrak{D}} e \veebar^{\mathfrak{D}} d) = (\sim^{\mathfrak{A}} \frac{1}{2}\|1) \in D^{\mathcal{D}} \ni e$. Hence, by (2.11) true in $\mathcal{D}$, we have $d \in D^{\mathcal{D}}$, in which case $J = \varnothing$, and so $\mathcal{D}$ is a subdirect I-power of $\mathcal{A}{\restriction}2$. Therefore, by (2.23), $\mathcal{D} \in \mathrm{Mod}(C^{\mathrm{PC}})$. In this way, $\mathsf{S} \subseteq \mathrm{Mod}(C^{\mathrm{PC}})$, in which case, for all $X \in \wp_\omega(\mathrm{Fm}^\omega_\Sigma)$, it holds that $C^{\mathrm{PC}}(X) \subseteq \mathrm{Cn}^\omega_{\mathsf{S}}(X) = C^{\mathrm{MP}}(X)$, and so C^{PC}, being finitary, for it is two-valued, is a sublogic of C^{MP}.

Thus, $C^{\mathrm{MP}} = C^{\mathrm{PC}}$ is consistent. Moreover, by Theorem 6.103, C^{NP} is defined by $\mathcal{L}_6$, in which (2.11) is not true under $[x_0/\langle\frac{1}{2},1\rangle, x_1/\langle 0,1\rangle]$. Finally, the following claim completes the argument:

Claim 6.105. *Any $\veebar$-disjunctive extension C' of C^{NP} is an extension of C^{MP}.*

Proof. In that case, we have $x_1 \in (C'(\{x_0, \sim x_0\}) \cap C'(\{x_0, x_1\})) = C'(\{x_0, \sim x_0 \veebar x_1\})$, as required. □

Next, by C^{DMP} we denote the extension of C relatively axiomatized by the *Dual Modens Ponens* rule:

$$\{\sim x_0, x_0 \veebar x_1\} \vdash x_1, \tag{6.20}$$

being actually dual to (2.11) for material implication. Clearly, by (2.8) satisfied in C, in view of its $\veebar$-disjunctivity, C^{DMP} is an extension of C^{NP}.

Lemma 6.106. *Suppose C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$; cf. Corollary 6.51). Then, the following hold:*

(i) *C^{DMP} is a proper extension of C^{NP};*
(ii) *$(\mathcal{A}{\restriction}2) \in \mathrm{Mod}(C^{\mathrm{DMP}})$;*
(iii) *providing $L_5 \triangleq (K_3 \cup M_2)$ forms a subalgebra of $\mathfrak{A}^2$, the following hold:*
  a) *$\sim^{\mathfrak{A}} \frac{1}{2} = 1 \leq^{\mathfrak{A}}_{\wedge} \frac{1}{2}$, that is, $\sim(x_0 \barwedge \sim x_0) \notin C(\varnothing)$;*
  b) *$\mathcal{A}$ is generated by $\{\frac{1}{2}\}$;*
  c) *$\mathcal{L}_6$ is generated by $L_6 \setminus L_5$;*
  d) *$\mathcal{A}$ is implicative;*
  e) *$\mathcal{L}_5 \triangleq (\mathcal{A}^2{\restriction}L_5) \in \mathrm{Mod}(C^{\mathrm{DMP}})$;*
  f) *the logic of $\mathcal{L}_5$ is an axiomatically-equivalent to C (and so proper) sublogic of C^{PC}, and so is its sulogic C^{DMP}.*

Proof. (i) Then, by Theorem 6.103, C^{NP} is defined by $\mathcal{L}_6$, in which (6.20) is not true under $[x_0/\langle\frac{1}{2},0\rangle, x_1/\langle 0,1\rangle]$, for $\mathcal{A}$ is both $\veebar$-disjunctive and $\sim$-paraconsistent.

(ii) Since $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} i = i$, for all $i \in 2$, the Σ-rule $(x_0 \veebar x_1) \vdash (\sim\sim x_0 \veebar x_1)$ is true in $\mathcal{A}{\restriction}2$, and so is (6.20), for (2.11) for the material implication is so, in view of Theorem 6.104.

(iii) **a)** If it did hold that $(\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2})|(\frac{1}{2} \leq^{\mathfrak{A}}_{\wedge} 1)$, then we would have $(\sim^{\mathfrak{A}^2}\langle\frac{1}{2},1\rangle | (\langle\frac{1}{2},1\rangle \barwedge^{\mathfrak{A}^2} \langle 1,0\rangle)) = \langle\frac{1}{2},0\rangle \notin L_5$, in which case $L_5 \supseteq \{\langle\frac{1}{2},1\rangle, \langle 1,0\rangle\}$ would not form a subalgebra of $\mathfrak{A}^2$, and so the $\sim$-paraconsistency of $\mathcal{A}$ and the linearity of the poset $\langle A, \leq^{\mathfrak{A}}_{\wedge}\rangle$ complete the argument.
  b) Then, by **a)**, we have $(\sim^{\mathfrak{A}})^{2-i}\frac{1}{2} = i$, for all $i \in 2$.
  c) Likewise, by **a)**, we have $(\sim^{\mathfrak{A}^2})^{2-i}\langle\frac{1}{2},0\rangle = \langle i,i\rangle$, for all $i \in 2$, while $(\langle\frac{1}{2},0\rangle \veebar^{\mathfrak{A}^2} \langle 1,1\rangle) = \langle\frac{1}{2},1\rangle$, whereas $(\sim^{\mathfrak{A}^2})^{2-i}\langle\frac{1}{2},1\rangle = \langle i,1-i\rangle$, for all $i \in 2$.
  d) Then, as $(L_6 \setminus L_5) \subseteq K_{3,0}$, by **c)**, we have $K'_{3,0} \supseteq L_6 \ni \langle 0,1\rangle$, and so Lemma 6.69**d)**⇒**e)** completes the argument.

e) Then, by (ii), (6.20) is true in $(\mathcal{A}{\restriction}2)^2 = (\mathcal{L}_5{\restriction}\Delta_2)$, while $(L_5 \setminus \Delta_2) = \{\langle\frac{1}{2},1\rangle\} \subseteq D^{\mathcal{L}_5}$, whereas, by **a)**, $\sim^{\mathcal{L}_5}\langle\frac{1}{2},1\rangle = \langle 1,0\rangle \notin D^{\mathcal{L}_5}$, in which case (6.20) is true in $\mathcal{L}_5$, and so (2.23), due to which $\mathcal{L}_5$ is a model of C, for $\mathcal{A}^2$ is so, completes the argument.

f) As $\mathcal{L}_5$ is both consistent and truth-non-empty, by **e)**, the logic of it is an inferentially consistent extension of C. Moreover, $(\pi_0{\restriction}L_5) \in \mathrm{hom}_\mathrm{S}(\mathcal{L}_5,\mathcal{A})$. In this way, **d)**, **e)**, (2.24), Corollaries 6.29, 6.47, Theorem 6.64, Remark 6.74 and Lemma 6.69**e)**⇒**b)** complete the argument. □

Lemma 6.107. *Let C' be an extension of C and $\mathcal{L}'_5$ the submatrix of $\mathcal{A}^2$ generated by L_5. Suppose C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$, C^{PC} being defined by $\mathcal{A}{\restriction}2$; cf. Corollary 6.51), while* (2.11) *is not satisfied in C'. Then, $\mathcal{L}'_5 \in \mathrm{Mod}(C')$. In particular, $C^{\mathrm{DMP}} = C^{\mathrm{PC}}$, unless L_5 forms a subalgebra of $\mathfrak{A}^2$.*

Proof. Then, by Theorem 6.103, C^{NP} is defined by $\mathcal{L}_6$. On the other hand, as C' does not satisfy the finitary (2.11), by Theorem 2.13, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of it not being a model of (2.11), in which case there are some $a \in D^{\mathcal{D}} \subseteq \{\frac{1}{2},1\}^I$ and some $b \in (D \setminus D^{\mathcal{D}})$ such that $(\sim^{\mathfrak{D}} a \veebar^{\mathfrak{D}} b) \in D^{\mathcal{D}}$, and so $J \triangleq \{i \in I \mid \pi_i(a) = \frac{1}{2}\} \supseteq K \triangleq \{i \in I \mid \pi_i(b) = 0\} \neq \varnothing$. Put $L \triangleq \{i \in I \mid \pi_i(b) = 1\}$. Then, given any $\bar{a} \in A^5$, set $(a_0\|a_1\|a_2\|a_3\|a_4) \triangleq ((((I \setminus (L \cup K)) \cap J) \times \{a_0\}) \cup ((I \setminus (L \cup J)) \times \{a_1\}) \cup ((L \setminus J) \times \{a_2\}) \cup ((L \cap J) \times \{a_3\}) \cup (K \times \{a_4\})) \in A^I$. In this way:

$$D \ni a = (\tfrac{1}{2}\|1\|1\|\tfrac{1}{2}\|\tfrac{1}{2}), \tag{6.21}$$

$$D \ni b = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|1\|0). \tag{6.22}$$

Moreover, by Claim 6.39, we also have:

$$D \ni f \triangleq (0\|0\|0\|0\|0), \tag{6.23}$$

$$D \ni t \triangleq (1\|1\|1\|1\|1). \tag{6.24}$$

Consider the following exhaustive (as $\sim^{\mathfrak{A}}\frac{1}{2} \in D^{\mathcal{A}} = \{\frac{1}{2},1\}$) cases:

- $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$.
  Then, in case $\frac{1}{2} \leq^{\mathfrak{A}}_{\barwedge} 1$, by (6.21) and (6.22), we have:

$$D \ni e \triangleq (a \barwedge^{\mathfrak{D}} b) = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|\tfrac{1}{2}\|0), \tag{6.25}$$

$$D \ni \sim^{\mathfrak{D}} e = (\tfrac{1}{2}\|\tfrac{1}{2}\|0\|\tfrac{1}{2}\|1), \tag{6.26}$$

$$D \ni c \triangleq (e \veebar^{\mathfrak{D}} \sim^{\mathfrak{D}} b) = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|\tfrac{1}{2}\|1), \tag{6.27}$$

$$D \ni \sim^{\mathfrak{D}} c = (\tfrac{1}{2}\|\tfrac{1}{2}\|0\|\tfrac{1}{2}\|0). \tag{6.28}$$

  Likewise, in case $\frac{1}{2}(\leq_{\barwedge} / \geq)^{\mathfrak{A}} 1$, by "(6.21) and (6.25)"/(6.22), we have:

$$D \ni d \triangleq ((e/b) \veebar^{\mathfrak{D}} \sim^{\mathfrak{D}} a) = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|\tfrac{1}{2}\|\tfrac{1}{2}), \tag{6.29}$$

$$D \ni \sim^{\mathfrak{D}} d = (\tfrac{1}{2}\|\tfrac{1}{2}\|0\|\tfrac{1}{2}\|\tfrac{1}{2}). \tag{6.30}$$

  Consider the following complementary subcases:
  - $L \subseteq J$.
    Then, since $I \supseteq K \neq \varnothing = (L \setminus J)$, by (6.23), (6.24) and (6.29), $\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{L}'_5$.
  - $L \nsubseteq J$.
    Then, consider the following complementary subsubcases:

* there is some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2},0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$,
  in which case, by (6.23) and (6.30), we have:
  $$D \ni \varphi^{\mathfrak{D}}(\sim^{\mathfrak{D}} d, f) = (0\|0\|1\|0\|0), \tag{6.31}$$
  $$D \ni \sim^{\mathfrak{D}} \varphi^{\mathfrak{D}}(\sim^{\mathfrak{D}} d, f) = (1\|1\|0\|1\|1). \tag{6.32}$$
  Then, since $(L \setminus J) \neq \varnothing \neq K$, taking (6.23), (6.24), (6.29), (6.30), (6.31) and (6.32) into account, we see that $\{\langle\langle g,h\rangle,(g\|g\|h\|g\|g)\rangle \mid \langle g,h\rangle \in L_6\}$ is an embedding of $\mathcal{L}_6$ into $\mathcal{D}$, and so, by (2.23), $\mathcal{L}_6$ is a model of C', for $\mathcal{D}$ is so, and so is its submatrix $\mathcal{L}'_5$, for $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.
* there is no $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(\frac{1}{2},0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$,
  Then, $\frac{1}{2} \leq^{\mathfrak{A}}_{\barwedge} 1$, for, otherwise, we would have $1 \leq^{\mathfrak{A}}_{\barwedge} \frac{1}{2}$, in which case we would get $\varphi^{\mathfrak{A}}(\frac{1}{2},0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$, where $\varphi \triangleq \sim(x_0 \barwedge \sim x_1) \in \mathrm{Fm}^2_\Sigma$. Consider the following complementary subsubsubcases:
  · $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) = \varnothing$.
    Then, taking (6.25), (6.26), (6.27), (6.28), (6.29) and (6.30) into account, as $K \neq \varnothing \neq (L \setminus J)$, we conclude that $\{\langle\langle g,h\rangle, (\frac{1}{2}\|\frac{1}{2}\|h\|\frac{1}{2}\|g)\rangle \mid \langle g,h\rangle \in L_6\}$ is an embedding of $\mathcal{L}_6$ into $\mathcal{D}$, and so, by (2.23), $\mathcal{L}_6$ is a model of C', for $\mathcal{D}$ is so, and so is its submatrix $\mathcal{L}'_5$, for $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.
  · $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing$.
    Let $\mathfrak{G}$ be the subalgebra of $\mathfrak{L}_6 \times \mathfrak{A}$ generated by $((L_6 \times \{\frac{1}{2}\}) \cup \{\langle\langle i,i\rangle,i\rangle \mid i \in 2\})$. Then, as $(((I \setminus (L \cup K)) \cap J) \cup (I \setminus (L \cup J)) \cup (L \cap J)) \neq \varnothing \notin \{K, L \setminus J\}$, by (6.23), (6.24), (6.25), (6.26), (6.27), (6.28), (6.29) and (6.30), we see that $\{\langle\langle\langle g,h\rangle,j\rangle,(j\|j\|h\|j\|g)\rangle \mid \langle\langle g,h\rangle,j\rangle \in G\}$ is an embedding of $\mathcal{G} \triangleq ((\mathcal{L}_6 \times \mathcal{A}){\upharpoonright}G)$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{G}$ is a model of C', for $\mathcal{D}$ is so. Let us prove, by contradiction, that $((D^{\mathcal{L}_6} \times \{0\}) \cap G) = \varnothing$. For suppose $((D^{\mathcal{L}_6} \times \{0\}) \cap G) \neq \varnothing$. Then, there is some $\psi \in \mathrm{Fm}^8_\Sigma$ such that $\psi^{\mathfrak{A}}(1,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0) = 0$ and $\psi^{\mathfrak{A}}(1,1,1,1,0,0,0,0) = 1$, for $\pi_1[D^{\mathcal{L}_6}] = \{1\}$. Let $\varphi \triangleq \psi(\sim x_1, \sim x_0, \sim x_0, \sim x_0, x_0, x_0, x_0, x_1) \in \mathrm{Fm}^2_\Sigma$. Then, $\varphi^{\mathfrak{A}}(\frac{1}{2},0) = 0$ and $\varphi^{\mathfrak{A}}(0,0) = 1$. This contradiction shows that $((D^{\mathcal{L}_6} \times \{0\}) \cap G) = \varnothing$, in which case $(\pi_0{\upharpoonright}G) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{G},\mathcal{L}_6)$, and so, by (2.23), $\mathcal{L}_6$ is a model of C', for $\mathcal{G}$ is so, and so is its submatrix $\mathcal{L}'_5$, for $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.

• $\sim^{\mathfrak{A}}\frac{1}{2} = 1$,
Consider the following exhaustive (as $\langle A, \leq^{\mathfrak{A}}_{\barwedge}\rangle$ is a chain poset) subcases:
– $\frac{1}{2} \leq^{\mathfrak{A}}_{\barwedge} 1$.
  Then, by (6.21) and (6.22), we get:
  $$D \ni c' \triangleq (a \veebar^{\mathfrak{D}} b) = (\tfrac{1}{2}\|1\|1\|1\|\tfrac{1}{2}), \tag{6.33}$$
  $$D \ni d' \triangleq \sim^{\mathfrak{D}} c' = (1\|0\|0\|0\|1), \tag{6.34}$$
  $$D \ni e' \triangleq \sim^{\mathfrak{D}} d' = (0\|1\|1\|1\|0), \tag{6.35}$$

$$D \ni f' \triangleq (c' \mathbin{\overline{\wedge}^{\mathfrak{D}}} d') = (\tfrac{1}{2}\|0\|0\|0\|\tfrac{1}{2}). \tag{6.36}$$

Consider the following complementary subsubcases:

* $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) = \varnothing$.
  Then, since $I \supseteq K \neq \varnothing$, by (6.23), (6.24) and (6.33), we see that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{L}'_5$.
* $((I \setminus (L \cup J)) \cup (L \setminus J) \cup (L \cap J)) \neq \varnothing$.
  Then, as $K \neq \varnothing$, by (6.23), (6.24), (6.33), (6.34), (6.35) and (6.36), we conclude that $\{\langle\langle g, h\rangle, (g\|h\|h\|h\|g)\rangle \mid \langle g, h\rangle \in L_6\}$ is an embedding of $\mathcal{L}_6$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{L}_6$ is a model of C', for $\mathcal{D}$ is so, and so is its submatrix $\mathcal{L}'_5$, for $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$.

– $1 \leq^{\mathfrak{A}}_{\overline{\wedge}} \frac{1}{2}$.
  Then, by (6.21) and (6.22), we get:

$$D \ni c'' \triangleq (a \mathbin{\underline{\vee}^{\mathfrak{D}}} b) = (\tfrac{1}{2}\|\tfrac{1}{2}\|1\|\tfrac{1}{2}\|\tfrac{1}{2}), \tag{6.37}$$

$$D \ni d'' \triangleq {\sim^{\mathfrak{D}}} c'' = (1\|1\|0\|1\|1), \tag{6.38}$$

$$D \ni e'' \triangleq {\sim^{\mathfrak{D}}} d'' = (0\|0\|1\|0\|0). \tag{6.39}$$

Consider the following complementary subsubcases:

* $L \subseteq J$.
  Then, as $K \neq \varnothing = (L \setminus J)$, taking (6.23), (6.24) and (6.37) into account, we see that $\{\langle g, I \times \{g\}\rangle \mid g \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so, and so is $\mathcal{L}'_5$.
* $L \nsubseteq J$.
  Then, as $K \neq \varnothing \neq (L \setminus J)$, taking (6.23), (6.24), (6.37), (6.38) and (6.39) into account, we see that $\{\langle\langle g, h\rangle, (g\|g\|h\|g\|g)\rangle \mid \langle g, h\rangle \in L'_5\}$ is an embedding of $\mathcal{L}'_5$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{L}'_5$ is a model of C', for $\mathcal{D}$ is so.

In this way, Theorem 6.104 and Lemma 6.106(i,ii) complete the argument, for $\mathcal{L}'_5 = \mathcal{L}_6$, unless L_5 forms a subalgebra of $\mathfrak{A}^2$, because $(L_6 \setminus L_5) = \{\langle \frac{1}{2}, 0\rangle\}$ is a singleton, while $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, since 2 forms a subalgebra of $\mathfrak{A}$. □

Corollary 6.108. *Let C' be an extension of C. Suppose* (6.20) *is not satisfied in C'. Then, $C' \subseteq C^{\mathrm{NP}}$.*

Proof. The case, when C^{NP} is inconsistent, is evident. Now, assume C^{NP} is consistent. Then, by Theorem 6.63, C is $\sim$-subclassical (i.e., 2 forms a subalgebra of $\mathfrak{A}$, C^{PC} being defined by $\mathcal{A}{\restriction}2$; cf. Corollary 6.51), in which case, by Theorem 6.63, C^{NP} is defined by $\mathcal{L}_6$. Consider the following complementary cases:

- L_5 forms a subalgebra of $\mathfrak{A}^2$.
  Then, as C' does not satisfy the finitary (6.20), by Theorem 2.13, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$ and some subdirect product $\mathcal{D} \in \mathrm{Mod}(C')$ of it not being a model of (6.20), in which case there are some $a \in D$ and some $b \in (D \setminus D^{\mathcal{D}})$ such that $(a \mathbin{\underline{\vee}^{\mathfrak{D}}} b) \in D^{\mathcal{D}} \ni {\sim^{\mathfrak{D}}} a$, in which case $a \in \{\frac{1}{2}, 0\}^I$, and so $J \triangleq \{i \in I \mid \pi_i(a) = \frac{1}{2}\} \supseteq \{i \in I \mid \pi_i(b) = 0\} \neq \varnothing$. Then, given any $\bar{a} \in A^2$, set $(a_0\|a_1) \triangleq ((J \times \{a_0\}) \cup ((I \setminus J) \times \{a_1\})) \in A^I$. In this way, $D \ni a = (\frac{1}{2}\|0)$. Consider the following complementary subcases:
  – $J = I$,
  Then, $D \ni a = (I \times \{\frac{1}{2}\})$, in which case, as $I = J \neq \varnothing$, by Lemma 6.106(iii)**b)**, $\{\langle x, I \times \{x\}\rangle \mid x \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$, and

so, by (2.23), $\mathcal{A}$ is a model of C', for $\mathcal{D}$ is so. In this way, $C' \subseteq C \subseteq C^{\mathrm{NP}}$.
– $J \neq I$,
Then, as $J \neq \varnothing \neq (I \setminus J)$, by Lemma 6.106(iii)**c)**, $\{\langle\langle x,y\rangle,(x\|y)\rangle \mid \langle x,y\rangle \in L_6\}$ is an embedding of $\mathcal{L}_6$ into $\mathcal{D}$, in which case, by (2.23), $\mathcal{L}_6$ is a model of C', for $\mathcal{D}$ is so, and so $C' \subseteq C^{\mathrm{NP}}$.

- L_5 does not form a subalgebra of $\mathfrak{A}^2$.
Then, $\mathcal{L}'_5 = \mathcal{L}_6$, for $(L_6 \setminus L_5) = \{\langle \frac{1}{2}, 0\rangle\}$ is a singleton, while $L_6 \supseteq L_5$ forms a subalgebra of $\mathfrak{A}^2$, because 2 forms a subalgebra of $\mathfrak{A}$. And what is more, by Theorem 6.104 and Lemma 6.106(ii), we have $C^{\mathrm{DMP}} \subseteq C^{\mathrm{PC}} = C^{\mathrm{MP}}$, in which case (2.11) is not satisfied in C', and so, by Lemma 6.107, we get $C' \subseteq C^{\mathrm{NP}}$. □

Finally, by Lemmas 6.31, 6.30, 6.69, 6.106, 6.107, Corollaries 6.47, 6.51, 6.53, 6.108, Theorems 6.63, 6.103, 6.104 and Remark 2.9(i)**d)**, we eventually get:

Theorem 6.109. *Suppose C is [not] non-$\sim$-subclassical — i.e., 2 is [not] non-$\mathfrak{A}$-closed — and (not) non-implicative [i.e., (n)either $K_{3,0}$ (n)or K_4 forms a subalgebra of $\mathfrak{A}^2$, while L_5 is ({not}) non-$\mathfrak{A}^2$-closed ({whereas C^{DMP} is ⟨not⟩ defined by $\mathcal{L}_5$})]. Then, the following hold:*

(i) *[({⟨some of⟩})] extensions of C form the $(2[+2(\{+1\langle+1\rangle\})])$-element chain $C \subsetneq C^{\mathrm{NP}} = [\mathrm{Cn}^{\omega}_{\mathcal{L}_6} \subsetneq] C^{\mathrm{DMP}} = [(\{\langle\subsetneq\rangle\, \mathrm{Cn}^{\omega}_{\mathcal{L}_5} \subsetneq\})](C^{\mathrm{INP}} =)C^{\mathrm{MP}} = [C^{\mathrm{PC}} = \mathrm{Cn}^{\omega}_{\mathcal{A}\restriction 2} \subsetneq]\, \mathrm{IC}$ [({⟨others being simultaneously extensions of C^{DMP} and sublogics of $\mathrm{Cn}^{\omega}_{\mathcal{L}_5}$⟩})];*
(ii) *$C[\cup(C^{\mathrm{PC}}(\cap(C^{\mathrm{NP}}\{\cup\, \mathrm{Cn}^{\omega}_{\mathcal{L}_5}\})))]$ is the structural completion of C.*

6.2.3. *Self-extensionality of conjunctive logics.*

Lemma 6.110. *Let $\mathcal{B} \in \mathrm{Mod}(C, \mathfrak{A})$. Suppose C is $\bar{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so), and $\mathcal{B}$ is truth-non-empty and consistent. Then, $\mathcal{B}$ is $\sim$-super-classical.*

Proof. In that case, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, some Σ-matrix $\mathcal{E}$ and some $(h|g) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}|\mathcal{D}, \mathcal{E})$, in which case $\mathcal{D}$ is truth-non-empty and consistent, for $\mathcal{B}$ is so, and so $I \neq \varnothing$, while, by Claim 6.39, $\{I \times \{c\} \mid c \in 2\} \subseteq D$. Given any Σ-matrix $\mathcal{H}$, set $\mathcal{H}' \triangleq (\mathcal{H}\restriction\{\sim\})$. In this way, $\mathcal{D}'$ is a submatrix of $(\mathcal{A}')^I$, while $(h|g) \in \hom^{\mathrm{S}}_{\mathrm{S}}((\mathcal{B}|\mathcal{D})', \mathcal{E}')$. And what is more, 2 forms a subalgebra of $\mathfrak{A}'$. Then, as $I \neq \varnothing$, $e \triangleq \{\langle c, I \times \{c\}\rangle \mid c \in 2\}$ is an embedding of $\mathcal{C} \triangleq (\mathcal{A}'\restriction 2)$ into $(\mathcal{A}')^I$, and so into $\mathcal{D}'$, for $(\mathrm{img}\, e) \subseteq D$, in which case, by (2.23), $\mathcal{C}$ is a $\sim$-classical model of the logic of $\mathcal{B}'$ (viz., the $\sim$-fragment of the logic of $\mathcal{B}$). In this way, Theorem 6.24 completes the argument. □

Since $\mathcal{A}_{\frac{1}{2}|(0+1)}$ is not $\sim$-super-classical, because, for all $a \in ((A \setminus D^{\mathcal{A}_{\frac{1}{2}}})|D^{\mathcal{A}_{0+1}}) = 2$, $\sim^{\mathfrak{A}} a \in 2$, by Lemma 6.110, we first have:

Corollary 6.111. *Suppose C is $\bar{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so). Then, $\mathcal{A}_{\frac{1}{2}|(0+1)} \notin \mathrm{Mod}(C)$.*

Lemma 6.112. *Let $\mathcal{B}$ be a three-valued $\sim$-super-classical $\bar{\wedge}$-conjunctive [and $\underline{\vee}$-disjunctive] Σ-matrix. Suppose $\mathfrak{B}$ is a $\bar{\wedge}$-semi-lattice [resp. $(\bar{\wedge}, \underline{\vee})$-lattice], in which case it is that with zero [and unit], for it is finite. Then, $\flat^{\mathfrak{B}}_{\bar{\wedge}[|\underline{\vee}]} \notin [| \in] D^{\mathcal{B}}$.*

Proof. In that case, by the $\bar{\wedge}$-conjunctivity[|$\underline{\vee}$-disjunctivity] of $\mathcal{B}$, since $(0[|1]) \notin [| \in] D^{\mathcal{B}}$, we have $\flat^{\mathfrak{B}}_{\bar{\wedge}[|\underline{\vee}]} = ((0[|1])(\bar{\wedge}[|\underline{\vee}])^{\mathfrak{B}} \flat^{\mathfrak{B}}_{\bar{\wedge}[|\underline{\vee}]}) \notin [| \in] D^{\mathcal{B}}$, as required. □

Corollary 6.113. *Suppose C is $\bar{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so), not $\sim$-classical (i.e., $\mathcal{A}$ is simple; cf. Corollary 6.36) and self-extensional, in which case $\mathfrak{A}$, being finite, is a $\bar{\wedge}$-semi-lattice (cf. Theorem 5.5(i)⇒(iv)) with zero. Then, the following hold:*

(i) $\frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$;
(ii) $(0 \mathbin{\overline{\wedge}}^{\mathfrak{A}} 1) = \flat^{\mathfrak{A}}_{\overline{\wedge}}$.

Proof. (i) By contradiction. For suppose $\frac{1}{2} \not\leq^{\mathfrak{A}}_{\overline{\wedge}} 1$. Then, $\flat^{\mathfrak{A}}_{\overline{\wedge}} \neq \frac{1}{2}$, in which case, by Lemma 6.112, $\flat^{\mathfrak{A}}_{\overline{\wedge}} = 0$, for $1 \in D^{\mathcal{A}}$, and so $0 \leq^{\mathfrak{A}}_{\overline{\wedge}} \frac{1}{2}$. Hence, $\frac{1}{2} \not\leq^{\mathfrak{A}}_{\overline{\wedge}} 0$, in which case $\mathfrak{A}_{\frac{1}{2}}$ is $\overline{\wedge}$-conjunctive, and so, by Theorem 5.5(i)⇒(iv), $\mathfrak{A}_{\frac{1}{2}}$, being truth-non-empty, is a model of C. This contradicts to Corollary 6.111.

(ii) In case $\mathcal{A}$ is false-singular, by Lemma 6.112, $\flat^{\mathfrak{A}}_{\overline{\wedge}} = 0$, and so $0 \leq^{\mathfrak{A}} 1$, that is, $(0 \mathbin{\overline{\wedge}}^{\mathfrak{A}} 1) = 0$. Otherwise, $\mathcal{A}$ is truth-singular, and so it (viz., C) is not $\sim$-paraconsistent, in which case, by the $\overline{\wedge}$-conjunctivity of C, $((x_0 \overline{\wedge} {\sim} x_0) \overline{\wedge} x_1) \equiv_C (x_0 \mathbin{\overline{\wedge}} {\sim} x_0)$, and so, as ${\sim}^{\mathfrak{A}} 0 = 1$, Corollary 3.5, the self-extensionality of C and the simplicity of $\mathcal{A}$ complete the argument. □

Theorem 6.114. *Suppose C is $\overline{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so) and not $\sim$-classical (i.e., $\mathcal{A}$ is [hereditarily] simple; cf. Corollary 6.36). Then, it is self-extensional iff, for all distinct $a, b \in A$, there is some $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(a)) \neq \chi^{\mathcal{A}}(h(b))$.*

Proof. The "if" part is by Theorem 5.1(v)⇒(i) with $\mathsf{C} = \{\mathfrak{A}\}$. Conversely, assume C is self-extensional. Consider any $\bar{a} \in (A^2 \setminus \Delta_A)$. Then, in case $\chi^{\mathcal{A}}(a_0) \neq \chi^{\mathcal{A}}(a_1)$, it suffices to take $h = \Delta_A \in \hom(\mathfrak{A}, \mathfrak{A})$. Now assume $\chi^{\mathcal{A}}(a_0) = \chi^{\mathcal{A}}(a_1)$, in which case there is some $j \in 2$ such that $a_j = \frac{1}{2}$ and $a_{1-j} = (1/0)$, whenever $\mathcal{A}$ is false-/truth-singular. Then, by Theorem 5.5(i)⇒(iv), $\mathfrak{A}$ is a $\overline{\wedge}$-semi-lattice, in which case, by the commutativity identity for $\overline{\wedge}$, $a_i \neq (a_i \mathbin{\overline{\wedge}}^{\mathfrak{A}} a_{1-i})$, for some $i \in 2$, and so $a_{1-i} \notin F \triangleq \{b \in A \mid a_i \leq^{\mathfrak{A}}_{\overline{\wedge}} b\} \ni a_i$. Hence, $\mathcal{B} \triangleq \langle \mathfrak{A}, F \rangle$ is a truth-non-empty consistent $\overline{\wedge}$-conjunctive Σ-matrix, in which case, by Theorem 5.5(i)⇒(iv), it is a model of C, and so, by Lemma 6.110, it is $\sim$-super-classical. Then, by Lemma 2.12, there are some finite set I, some $\overline{\mathcal{C}} \in \mathbf{S}_*(\mathcal{A})^I$, some subdirect product $\mathcal{D}$ of it, some Σ-matrix $\mathcal{E}$ and some $(f|g) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{B}|\mathcal{D}, \mathcal{E})$, in which case $\mathcal{D}$ is truth-non-empty and consistent, for $\mathcal{B}$ is so, and so $I \neq \varnothing$, while, by Claim 6.39, $\{I \times \{c\} \mid c \in 2\} \subseteq D$. Consider the following exhaustive cases:

- $\mathcal{A}$ is false-singular.
  Then, by Lemma 6.112 and Corollary 6.113(i), we have $0 = \flat^{\mathfrak{A}}_{\overline{\wedge}} \leq^{\mathfrak{A}}_{\overline{\wedge}} \frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} 1$, in which case $i = (1 - j)$, and so $\mathcal{B} = \mathcal{A}_1$ is truth-singular, and so is $\mathcal{E}$. Consider the following complementary subcases:
  - ${\sim}^{\mathfrak{A}} \frac{1}{2} = 0$.
    Then, $\mathcal{A}$ is $\sim$-negative, in which case, by Remark 2.9(i)**a)**, it, being $\overline{\wedge}$-conjunctive, is $\overline{\wedge}^{\sim}$-disjunctive, and so, by Theorem 5.5(i)⇒(iv), $\mathfrak{A}$ is a $(\overline{\wedge}, \overline{\wedge}^{\sim})$-lattice. Hence, $\mathcal{B}$, being $\overline{\wedge}$-conjunctive, is $\overline{\wedge}^{\sim}$-disjunctive. Therefore, by Corollary 3.20 and Remark 2.8(ii), there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$.
  - ${\sim}^{\mathfrak{A}} \frac{1}{2} \neq 0$.
    Then, $\frac{1}{2} \leq^{\mathfrak{A}}_{\overline{\wedge}} {\sim}^{\mathfrak{A}} \frac{1}{2}$, in which case $(\frac{1}{2} \mathbin{\overline{\wedge}}^{\mathfrak{A}} {\sim}^{\mathfrak{A}} \frac{1}{2}) = \frac{1}{2}$, and so ${\sim}^{\mathfrak{A}}(\frac{1}{2} \mathbin{\overline{\wedge}}^{\mathfrak{A}} {\sim}^{\mathfrak{A}} \frac{1}{2}) = {\sim}^{\mathfrak{A}} \frac{1}{2} \in D^{\mathcal{A}}$. Moreover, for each $k \in 2$, $(k \mathbin{\overline{\wedge}}^{\mathfrak{A}} {\sim}^{\mathfrak{A}} k) = 0$, in which case ${\sim}^{\mathfrak{A}}(k \mathbin{\overline{\wedge}}^{\mathfrak{A}} {\sim}^{\mathfrak{A}} k) = 1 \in D^{\mathcal{A}}$, and so ${\sim}(x_k \mathbin{\overline{\wedge}} {\sim} x_k) \in C(\varnothing)$. Therefore, by Lemma 5.7, $1 = {\sim}^{\mathfrak{A}}(1 \mathbin{\overline{\wedge}}^{\mathfrak{A}} {\sim}^{\mathfrak{A}} 1) = {\sim}^{\mathfrak{A}}(\frac{1}{2} \mathbin{\overline{\wedge}}^{\mathfrak{A}} {\sim}^{\mathfrak{A}} \frac{1}{2}) = {\sim}^{\mathfrak{A}} \frac{1}{2}$. Hence, $\mathcal{B}$ is $\sim$-negative, in which case, by Remark 2.9(i)**a,c)**, it, being $\overline{\wedge}$-conjunctive, is $\sqsupset^{\sim}_{\overline{\wedge}\sim}$-implicative. Consider the following complementary subsubcases:
    - $\mathcal{B}$ is simple.
      Then, by Lemma 6.33(iii)⇒(ii), it is hereditarily simple, so, by

Theorem 3.11(i)⇒(iii) and Remark 5.13(iv), it, being implicative, has an axiomatic binary equality determinant ε. Moreover, if 2 would not form a subalgebra of $\mathfrak{A}$, then there would be some $\varphi \in \mathrm{Fm}^2_\Sigma$ such that $\varphi^{\mathfrak{A}}(0,1) = \frac{1}{2}$, in which case we would have $D \ni \varphi^{\mathfrak{D}}(I \times \{0\}, I \times \{1\}) = (I \times \{\frac{1}{2}\})$, and so, as $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in A\}$ would be an embedding of $\mathcal{A}$ into $\mathcal{D}$, and so, by Remark 2.8(ii), $g \circ e$ would be that into the truth-singular $\mathcal{E}$, contrary to the fact that $\mathcal{A}$, being false-singular, is not truth-singular. Thus, 2 does form a subalgebra of $\mathfrak{A}$, in which case, by Lemma 3.10, ε is an equality determinant for $(\mathcal{B}{\restriction}2) = \mathcal{C} \triangleq (\mathcal{A}{\restriction}2)$. And what is more, by Remark 2.8, f is injective, in which case $g' \triangleq (f^{-1} \circ g) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{D}, \mathcal{B})$, and so there is some $a \in (D \setminus D^{\mathcal{D}})$ such that $g'(a) = \frac{1}{2} \notin (B \setminus D^{\mathcal{B}})$, in which case there is some $l \in I$ such that $\pi_l(a) = 0$. Let $\mathcal{F}$ be the submatrix of $\mathcal{D}$ generated by $\{a\}$, in which case $h' \triangleq (\pi_l{\restriction}F) \in \hom(\mathcal{F}, \mathcal{C})$, for $h'(a) = 0 \in 2$, while $f' \triangleq (g'{\restriction}F) \in \hom^{\mathrm{S}}_{\mathrm{S}}(\mathcal{F}, \mathcal{B})$, for $\mathfrak{A}$ is generated by $\{\frac{1}{2}\} = f'[\{a\}]$, because every $m \in 2$ is equal to $(\sim^{\mathfrak{A}})^{2-m}\frac{1}{2}$, whereas, since ε is an axiomatic binary equality determinant for both $\mathcal{B}$ and $\mathcal{C}$, by (3.1), we also have $(\ker f') = f'^{-1}[\Delta_B] = f'^{-1}[\theta^{\mathcal{B}}_\varepsilon] = \theta^{\mathcal{F}}_\varepsilon \subseteq h'^{-1}[\theta^{\mathcal{C}}_\varepsilon] = h'^{-1}[\Delta_2] = (\ker h')$, and so, by the Homomorphism Theorem, $h \triangleq (h' \circ f'^{-1}) \in \hom(\mathcal{B}, \mathcal{C})$. Then, $h(\frac{1}{2}) = h'(a) = 0$, in which case $h(0) = h(\sim^{\mathfrak{A}}\sim^{\mathfrak{A}}\frac{1}{2}) = \sim^{\mathfrak{A}}\sim^{\mathfrak{A}}h(\frac{1}{2}) = \sim^{\mathfrak{A}}\sim^{\mathfrak{A}}0 = 0$, and so $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{C}) \subseteq \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$.

∗ $\mathcal{B}$ is not simple.
Then, by (2.23), Lemma 6.33(ii)⇒(i) and Corollary 6.51, 2 forms a subalgebra of $\mathfrak{A}$, while there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A}{\restriction}2) \subseteq \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$.

- $\mathcal{A}$ is truth-singular.
  Then, so is $\mathcal{E}$, while, by Lemma 6.112, we have the following exhaustive subcases:
  - $\flat^{\mathfrak{A}}_{\bar\wedge} = 0$.
    Then, $i = j$, in which case, by Corollary 6.113(i), $\mathcal{B} = \mathcal{A}_{1+}$, and so $f(1)$ and $f(\frac{1}{2})$, being distinguished values of $\mathcal{E}$, for both 1 and $\frac{1}{2}$ are those of $\mathcal{B}$, are equal, for $\mathcal{E}$ is truth-singular. Hence, f is not injective, in which case, by Remark 2.8, $\mathcal{B}$ is not simple, and so, by (2.23), Lemma 6.33(ii)⇒(i) and Corollary 6.51 and (2.23), 2 forms a subalgebra of $\mathfrak{A}$, while there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A}{\restriction}2) \subseteq \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$.
  - $\flat^{\mathfrak{A}}_{\bar\wedge} = \frac{1}{2}$.
    Then, by Corollary 6.113(ii), $(0 \mathbin{\bar\wedge^{\mathfrak{A}}} 1) = \frac{1}{2}$, in which case $D \ni ((I \times \{0\}) \mathbin{\bar\wedge^{\mathfrak{D}}} (I \times \{1\})) = (I \times \{\frac{1}{2}\})$, and so, as $I \neq \varnothing$, $e \triangleq \{\langle d, I \times \{d\}\rangle \mid d \in A\}$ is an embedding of $\mathcal{A}$ into $\mathcal{D}$. Then, by Remark 2.8(ii), $g' = (g \circ e)$ is an embedding of $\mathcal{A}$ into $\mathcal{E}$, in which case $3 = |A| \leqslant |E| \leqslant |B| = |A| = 3$, and so $|E| = 3 = |A|$. Therefore, $(\mathrm{img}\, g') = E$, because $3 \leqslant n$, for no $n \in 3$, and so $h \triangleq (g'^{-1} \circ f) \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$.

Thus, anyway, there is some $h \in \hom_{\mathrm{S}}(\mathcal{B}, \mathcal{A})$, in which case $h(a_{1-i}) \notin D^{\mathcal{A}} \ni h(a_i)$, and so $\chi^{\mathcal{A}}(h(a_i)) = 1 \neq 0 = \chi^{\mathcal{A}}(h(a_{1-i}))$, as required. □

Theorem 6.115. *Suppose both C is both $\bar\wedge$-conjunctive (viz., $\mathcal{A}$ is so) and not $\sim$-classical (i.e., $\mathcal{A}$ is simple; cf. Corollary 6.36), and $\mathcal{A}$ is false-/truth-singular. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) $h_{0/(1[-])} \in \hom(\mathfrak{A}, \mathfrak{A})$*;*
(iii) $\mathcal{A}_{1/(1+)|0)} \in \mathrm{Mod}(C)$.

Proof. First, assume (i) holds. Then, by Theorem 5.5(i)⇒(iv), $\mathfrak{A}$, being finite, is a $\bar{\wedge}$-semi-lattice with zero. Moreover, as $\frac{1}{2} \neq (1/0)$, by Theorem 6.114, there is an endomorphism h of $\mathfrak{A}$ such that $\chi^{\mathcal{A}}(h(\frac{1}{2})) \neq \chi^{\mathcal{A}}(h(1/0))$, in which case $h(\frac{1}{2}) \neq h(1/0)$, and so $B \triangleq (\operatorname{img} e)$ forms a non-one-element subalgebra of $\mathfrak{A}$. Hence, $2 \subseteq B$, while h is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{B} \triangleq (\mathfrak{A} \upharpoonright B)$. Then, by Lemma 6.112, we have the following two exhaustive cases:

- $\flat^{\mathfrak{A}}_{\bar{\wedge}} = 0 \in B$ (in particular, $\mathcal{A}$ is false-singular).
  Then, both $\mathfrak{A}$ and $\mathfrak{B}$ are $\bar{\wedge}$-semilattices with zero 0, in which case, by Lemma 2.3, $h(0) = 0 \notin D^{\mathcal{A}}$, and so $D^{\mathcal{A}} \ni 1 = \sim^{\mathfrak{A}} 0 = \sim^{\mathfrak{A}} h(0) = h(\sim^{\mathfrak{A}} 0) = h(1)$. Therefore, $h(\frac{1}{2}) \notin / \in D^{\mathcal{A}}$, in which case $h(\frac{1}{2}) = (0/1)$, and so $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_{0/1}$.
- $\flat^{\mathfrak{A}}_{\bar{\wedge}} = \frac{1}{2}$, in which case $\mathcal{A}$ is truth-singular.
  Then, by Corollary 6.113(ii), $\frac{1}{2} \in B \supseteq 2$, in which case $B = A$, and so, by Lemma 2.3, $h(\frac{1}{2}) = \frac{1}{2} \notin D^{\mathcal{A}}$. Hence, $h(0) \in D^{\mathcal{A}}$, in which case $h(0) = 1$, and so $0 = \sim^{\mathfrak{A}} 1 = \sim^{\mathfrak{A}} h(0) = h(\sim^{\mathfrak{A}} 0) = h(1)$. In this way, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_{1-}$.

Thus, (ii) holds. Next, (ii)⇒(iii) is by (2.23), (6.11) and (6.12). Finally, (iii)⇒(i) is by Theorem 5.1(vi)⇒(i), (6.9) and (6.10). □

Corollary 6.116. *Suppose C is $\bar{\wedge}$-conjunctive (viz., $\mathcal{A}$ is so), not $\sim$-classical (i.e., $\mathcal{A}$ is simple; cf. Corollary 6.36) and self-extensional, in which case $\mathfrak{A}$, being finite, is a $\bar{\wedge}$-semi-lattice (cf. Theorem 5.5(i)⇒(iv)) with zero. Then, the following are equivalent:*

(i) *C is $\sim$-subclassical;*
(ii) $\flat^{\mathfrak{A}}_{\bar{\wedge}} = 0$ *(in particular, $\mathcal{A}$ is false-singular; cf. Lemma 6.112);*
(iii) $\partial(\mathcal{A}) \in \mathrm{Mod}(C)$*;*
(iv) $h_{0/1} \in \hom(\mathfrak{A}, \mathfrak{A})$*, whenever $\mathcal{A}$ is false-/truth-singular;*
(v) $\mathcal{A}_0 \notin \mathrm{Mod}(C)$*;*
(vi) $h_{1-} \notin \hom(\mathfrak{A}, \mathfrak{A})$*;*
(vii) $\sim^{\mathfrak{A}} \frac{1}{2} \neq \frac{1}{2}$.

Proof. We use Corollary 6.51 tacitly. Then, as $1 \in D^{\mathcal{A}}$, (i)⇒(ii) is by Lemma 6.112 and Corollary 6.113(ii). Next, (iv)⇒(i) is by the fact that $(\operatorname{img} h_{0/1}) = 2$.

Now, assume (ii) holds. Then, $\mathcal{A}_0$ is not $\bar{\wedge}$-conjunctive, and so (v) holds, for C is $\bar{\wedge}$-conjunctive. Likewise, if h_{1-} was an endomorphism of $\mathfrak{A}$, then, as, by (ii), $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$, that is, $(0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = 0$, we would have $1 = h_{1-}(0) = h_{1-}(0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) = (h_{1-}(0) \mathbin{\bar{\wedge}^{\mathfrak{A}}} h_{1-}(1)) = (1 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 0)$, that is, $1 \leq^{\mathfrak{A}}_{\bar{\wedge}} 0$, in which case we would get $0 = 1$, and so (vi) holds.

Further, (vii)⇒(vi) is by the following claim:

Claim 6.117. *Suppose $h_{1-} \in \hom(\mathfrak{A}, \mathfrak{A})$. Then, $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$.*

Proof. If $\sim^{\mathfrak{A}} \frac{1}{2}$ was not equal to $\frac{1}{2}$, then it would be equal to some $i \in 2$, in which case we would have $(1 - i) = h_{1-}(i) = h_{1-}(\sim^{\mathfrak{A}} \frac{1}{2}) = \sim^{\mathfrak{A}} h_{1-}(\frac{1}{2}) = \sim^{\mathfrak{A}} \frac{1}{2} = i$. □

Likewise, if it did hold that $\sim^{\mathfrak{A}} \frac{1}{2} = \frac{1}{2}$, while $h_i \in \hom(\mathfrak{A}, \mathfrak{A})$, for some $i \in 2$, then we would have $i = h_i(\frac{1}{2}) = h_i(\sim^{\mathfrak{A}} \frac{1}{2}) = \sim^{\mathfrak{A}} h_i(\frac{1}{2}) = \sim^{\mathfrak{A}} i = (1 - i)$. Therefore, (iv)⇒(vii) holds.

Furthermore, (v/vi)⇒(iii/iv) is by Theorem 6.115(i)⇒(iii/ii), respectively.

Finally, we prove (iii)⇒(iv) by contradiction. For suppose (iii) holds, while (iv) does not hold. Then, by Theorem 6.115(i)⇒(ii), $\mathcal{A}$ is truth-singular, in which case, by (iii), $\mathcal{A}_{1+} = \partial(\mathcal{A}) \in \mathrm{Mod}(C)$, while $h_{1-} \in \hom(\mathfrak{A}, \mathfrak{A})$, in which case, by (2.23) and (6.12), $\mathcal{A}_{0+} \in \mathrm{Mod}(C)$, and so C is not $\bar{\wedge}$-conjunctive, for $\mathcal{A}_{0+}$ is not so, in view of Corollary 6.113(i). This contradiction completes the argument. □

Next, $\mathcal{A}$ is said to *have Dual Truth Closure Condition (DTCC) with respect to* $\bar{\wedge}$, provided $(a \mathbin{\bar{\wedge}^{\mathfrak{A}}} b) \in D^{\partial(\mathcal{A})}$, for all distinct $a, b \in D^{\partial(\mathcal{A})}$.

Corollary 6.118. *Suppose $\mathcal{A}$ is [both] $\bar{\wedge}$-conjunctive (viz., C is so) [and not $\sim$-negative, unless C is $\sim$-classical]. Then, C is both self-extensional and $\sim$-subclassical [if and] only if both C has PWC w.r.t. $\sim$ and either C is $\sim$-classical or both $\mathfrak{A}$ is a $\bar{\wedge}$-semilattice and $\mathcal{A}$ has DTCC w.r.t. $\bar{\wedge}$.*

Proof. First, assume C is both self-extensional and $\sim$-subclassical. Consider the following complementary cases:

- C is $\sim$-classical,
  in which case, by Remark 2.9(i)**b)**, C has PWC w.r.t. $\sim$.
- C is not $\sim$-classical.
  Then, by Corollaries 6.113(i) and 6.116(i)⇒(ii), $\mathfrak{A}$ is a $\bar{\wedge}$-semi-lattice with $0 \leq^{\mathfrak{A}}_{\bar{\wedge}} \frac{1}{2} \leq^{\mathfrak{A}}_{\bar{\wedge}} 1$, in which case $\mathcal{A}$ has DTCC w.r.t. $\bar{\wedge}$, while $\sim^{\mathfrak{A}}$ is anti-monotonic with respect to $\leq^{\mathfrak{A}}_{\bar{\wedge}}$, and so, by Theorem 5.5(i)⇒(ii), C has PWC w.r.t. $\sim$.

[Conversely, assume both C has PWC w.r.t. $\sim$ and either C is $\sim$-classical or both $\mathfrak{A}$ is a $\bar{\wedge}$-semi-lattice and $\mathcal{A}$ has DTCC w.r.t. $\bar{\wedge}$. Consider the following complementary cases:

- C is $\sim$-classical.
  Then, it is both $\sim$-subclassical and, by Example 5.2, self-extensional.
- C is not $\sim$-classical.
  Then, $\mathfrak{A}$ is a $\bar{\wedge}$-semi-lattice, while $\mathcal{A}$ has DTCC w.r.t. $\bar{\wedge}$ as well as is both non-$\sim$-negative and false-/truth-singular, in which case $\sim^{\mathfrak{A}}\frac{1}{2} \neq (0/1)$, and so $D^{\partial(\mathcal{A})} = (\sim^{\mathfrak{A}})^{-1}[A \setminus D^{\mathcal{A}}]$. Consider any $\phi \in \mathrm{Fm}^{\omega}_{\Sigma}$, any $\psi \in C(\phi)$, in which case $\sim\phi \in C(\sim\psi)$, and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h(\phi) \in D^{\partial(\mathcal{A})}$, in which case $h(\sim\phi) \notin D^{\mathcal{A}}$, and so $h(\sim\psi) \notin D^{\mathcal{A}}$, that is, $h(\psi) \in D^{\partial(\mathcal{A})}$. Thus, $\partial(\mathcal{A})$ is a $(2 \setminus 1)$-model of C. In particular, for each $i \in 2$, the unary Σ-rule $(x_0 \mathbin{\bar{\wedge}} x_1) \vdash x_i$, being satisfied in C, for this is $\bar{\wedge}$-conjunctive, is true in $\partial(\mathcal{A})$. Conversely, consider any $\bar{a} \in (D^{\partial(\mathcal{D})})^2$. Then, in case $a_0 = a_1$, by the idempotencity identity for $\bar{\wedge}$, we have $(a_0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} a_1) = a_0 \in D^{\partial(\mathcal{A})}$. Otherwise, since $\mathcal{A}$ has DTCC w.r.t. $\bar{\wedge}$, we have $(a_0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} a_1) \in D^{\partial(\mathcal{A})}$ too. Thus, $\partial(\mathcal{A})$ is $\bar{\wedge}$-conjunctive, in which case, by Lemma 5.3, it, being truth-non-empty, is a model of C, and so, by Theorem 6.115(iii)⇒(i) and Corollary 6.116(iii)⇒(i), C is both self-extensional and $\sim$-subclassical.] □

6.2.3.1. Self-extensionality of both conjunctive and disjunctive logics.

Lemma 6.119. *Suppose C is both $\bar{\wedge}$-conjunctive and $\veebar$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 6.30) as well as both non-$\sim$-classical (i.e., $\mathcal{A}$ is simple; cf. Corollary 6.36) and self-extensional. Then, $\mathfrak{A}$ is a distributive $(\bar{\wedge}, \veebar)$-lattice with zero* 0 *and unit* 1.

Proof. Then, by Theorem 5.5(i)⇒(iv), $\mathfrak{A}$, being finite, is a distributive $(\bar{\wedge}, \veebar)$-lattice with zero and unit, in which case, as $|A| = 3$, $\langle A, \leq^{\mathfrak{A}}_{\bar{\wedge}} \rangle$ is a chain, and so $(0 \mathbin{\bar{\wedge}^{\mathfrak{A}}} 1) \in 2$. In this way, as $1 \in D^{\mathcal{A}}$, Lemma 6.112 and Corollary 6.113 complete the argument. □

As for negative instances of Lemma 6.119, as a first one, we should like to highlight P^1 [26] (cf. [15]), in which case $\mathfrak{A}$ has no semi-lattice (even merely idempotent and commutative) secondary operations, simply because the values of primary ones belong to $2 \not\ni \frac{1}{2}$, in which case 2 forms a subalgebra of $\mathfrak{A}$, and so $\mathcal{A}$, being $\supset$-implicative, is both $\uplus_{\supset}$-disjunctive and $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \supset \sim(x_0 \supset x_0))$ (in particular, this is $\uplus_{\supset}^{\neg}$-conjunctive; cf. Remark 2.9(i)**a)**). Likewise, three-valued expansions of HZ [6] are not self-extensional, because, in that case, though $\mathcal{A}$, being false-singular, is neither $\wedge$-conjunctive nor $\vee$-disjunctive, simply because $\mathfrak{A}$ is a $(\wedge, \vee)$-lattice but with distinguished zero $\frac{1}{2}$, $\mathfrak{A}$ is a $(\vee^{\sim}, \wedge^{\sim})$-lattice with zero 0 and unit $\frac{1}{2} \neq 1$, in which case $\mathcal{A}$ is both $\vee^{\sim}$-conjunctive and $\wedge^{\sim}$-disjunctive. On the other hand, arbitrary three-valued expansions of both P^1 and HZ are covered by the next subsection as well, the latter ones being equally covered by the following characterization (more precisely, some of its consequences, as we show below):

Theorem 6.120. *Suppose C is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 6.30) as well as both C is not $\sim$-classical (i.e., $\mathcal{A}$ is simple; cf. Corollary 6.36) and $\mathcal{A}$ is false-/truth-singular. Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *$h_{0/1}$ is an endomorphism of $\mathfrak{A}$;*
(iii) *$\partial(\mathcal{A}) \in \mathrm{Mod}(C)$;*
(iv) *$\mathfrak{A}$ is a [distributive] $(\overline{\wedge}, \underline{\vee})$-lattice {with zero 0 and unit 1} having a non-singular non-diagonal (partial) endomorphism.*

Proof. First, (i)⇒(ii) is Corollary 6.116(ii)⇒(iv) and Lemma 6.119. Next, (ii)⇒(iii) is by (2.23) and (6.11). Further, (iii)⇒(i) is by (6.9) and Theorem 5.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}, \partial(\mathcal{A})\}$. Thus, we have proved the equivalence of (i,ii,iii). Furthermore, (i,ii)⇒(iv) is by Lemma 6.119 and the fact that $h_{0/1}(\frac{1}{2}) \in 2 \not\ni \frac{1}{2}$, while $(\mathrm{img}\, h_{0/1}) = 2$ is not a singleton.

Finally, assume (iv) holds. Then, there are some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and some non-diagonal non-singular $h \in \hom(\mathfrak{B}, \mathfrak{A})$, in which case $D \triangleq (\mathrm{img}\, h)$ forms a non-one-element subalgebra of $\mathfrak{A}$, and so does $B = (\mathrm{dom}\, h)$. Hence, $2 \subseteq (B \cap D)$, in which case, by Lemma 6.112, both $\mathfrak{B}$ and $\mathfrak{D} \triangleq (\mathfrak{A} \restriction D)$ are $(\overline{\wedge}, \underline{\vee})$-lattices with zero/unit 0/1, and so, as $h \in \hom(\mathfrak{B}, \mathfrak{D})$ is surjective, by Lemma 2.3, $h(0/1) = (0/1)$, in which case $(1/0) = \sim^{\mathfrak{A}}(0/1) = \sim^{\mathfrak{A}} h(0/1) = h(\sim^{\mathfrak{A}}(0/1)) = h(1/0)$, and so $h \restriction 2$ is diagonal. Therefore, $B = A$, while $h(\frac{1}{2}) \neq \frac{1}{2}$. In this way, if $h(\frac{1}{2})$ was equal to 1/0, then h would be a non-injective strict homomorphism from $\mathcal{A}$ to itself, in which case, by Remark 2.8(ii), $\mathcal{A}$ would not be simple. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_{0/1}$, so (ii) holds, as required. □

First, by Lemma 5.14 and Theorem 6.120(i)⇔(iv), we immediately have:

Corollary 6.121. *Suppose $\mathcal{A}$ is both $\overline{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., C is so; cf. Lemma 6.30) as well as either $\sim$-paraconsistent or $(\underline{\vee}, \sim)$-paracomplete (in which case C is so, and so is not $\sim$-classical, while $\{x_0, \sim x_0\}$ is a unary unitary equality determinant for $\mathcal{A}$). Then, C is self-extensional iff the following hold:*

(i) *$\mathcal{A}$ has no equational implication;*
(ii) *$\mathfrak{A}$ is a {distributive} $(\overline{\wedge}, \underline{\vee})$-lattice [with zero 0 and unit 1].*

In view of Theorems 10, 13 and Example 10 of [22], this positively covers [the implication-less fragment of] Gödel's three-valued logic [4] as well as their "$\sim$-paraconsistent counterparts" resulted from lattice duality — viz., using dual (relative) pseudo-complement(s) instead of the direct one(s). As to negative instances of Theorem 6.120, we need some its generic consequences.

First, by Corollary 6.116(ii)⇒(i) and Lemma 6.119, we immediately have:

Corollary 6.122. *Suppose C is both $\bar{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 6.30) as well as self-extensional. Then, C is $\sim$-subclassical.*

Then, by Corollaries 6.118, 6.122 and Lemmas 6.112 and 6.119, we get:

Corollary 6.123. *Suppose $\mathcal{A}$ is both $\bar{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., C is so; cf. Lemma 6.30) [as well as not $\sim$-negative (in particular, either $\sim$-paraconsistent or $(\underline{\vee},\sim)$-paracomplete $\{$viz., C is so$\}$), unless C is $\sim$-classical]. Then, C is self-extensional [if and] only if both C has PWC with respect to $\sim$ and either C is $\sim$-classical or $\mathfrak{A}$ is a $(\bar{\wedge},\underline{\vee})$-lattice.*

Likewise, by Corollaries 6.116(i)⇒(vii), 6.122, Remark 2.9(ii), Lemma 6.33(ii)⇒(i) and Corollary 6.36, we also get:

Corollary 6.124. *Suppose C is both $\bar{\wedge}$-conjunctive and $\underline{\vee}$-disjunctive (viz., $\mathcal{A}$ is so; cf. Lemma 6.30) as well as self-extensional. Then, $\sim^{\mathfrak{A}}\frac{1}{2} \neq \frac{1}{2}$.*

These negatively cover arbitrary three-valued expansions (cf. Corollary 6.40 in this connection) of both Kleene's three-valued logic [7] (including those of Łukasiewicz' one L_3 [9]) and LP [14] (including those of the *logic of antinomies* LA [1]) as well as of HZ. On the other hand, three-valued expansions of Ł$_3$, LA and HZ are equally covered by the next subsubsection.

The condition of the $\underline{\vee}$-disjunctivity of $\mathcal{A}/C$ can not be omitted in the formulations of Corollaries 6.122 and 6.124, as it is demonstrated by:

Example 6.125. Let $\mathcal{A}$ be truth-singular, $\Sigma \triangleq \{\wedge,\sim\}$, $\sim^{\mathfrak{A}} \triangleq h_{1-}$ and $\wedge^{\mathfrak{A}} \triangleq ((\pi_0{\restriction}\Delta_A) \cup ((A \setminus \Delta_2) \times \{\frac{1}{2}\}))$. Then, $\mathcal{A}$ is $\wedge$-conjunctive, while $\langle\sim^{\mathfrak{A}}0, \sim^{\mathfrak{A}}\frac{1}{2}\rangle = \langle 1, \frac{1}{2}\rangle \notin \theta^{\mathcal{A}} \ni \langle 0, \frac{1}{2}\rangle$, in which case $\theta^{\mathcal{A}} \notin \mathrm{Con}(\mathfrak{A})$, whereas $(0 \wedge^{\mathfrak{A}} 1) = \frac{1}{2} \notin 2$, in which case 2 does not form a subalgebra of $\mathfrak{A}$, and so, by Theorem 6.35(i)⇒(v) [and Corollary 6.51], C is not $\sim$-[sub]classical. On the other hand, $h_{1-} \in \hom(\mathfrak{A},\mathfrak{A})$, so by Theorem 6.115(ii)⇒(i), C is self-extensional. □

6.2.4. *Self-extensionality of implicative logics.*

Lemma 6.126. *Suppose C is $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 6.31) and not $\sim$-classical. Then, $h_i \in \hom(\mathfrak{A},\mathfrak{A})$, for no $i \in 2$.*

Proof. By contradiction. For suppose $h_i \in \hom(\mathfrak{A},\mathfrak{A})$, for some $i \in 2$, in which case $(\ker h_i) \in \mathrm{Con}(\mathfrak{A})$, and so, if i was equal to 1/0, whenever $\mathcal{A}$ was false-/truth-singular, then $\theta^{\mathcal{A}}$ would be equal to $(\ker h_i) \in \mathrm{Con}(\mathfrak{A})$, contrary to Theorem 6.35(v)⇒(i), while $2 = (\mathrm{img}\, h_i)$ forms a subalgebra of $\mathfrak{A}$, and so $((0/1) \sqsupset^{\mathfrak{A}} 0) = (1/0)$, whenever $\mathcal{A}$ is false-/truth-singular. Therefore, $i = (0/1)$, whenever $\mathcal{A}$ is false-/truth-singular, in which case $(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = (0/1)$, and so $(0/1) = h_i(0/1) = h_i(\frac{1}{2} \sqsupset^{\mathfrak{A}} 0) = (h_i(\frac{1}{2}) \sqsupset^{\mathfrak{A}} h_i(0)) = ((0/1) \sqsupset^{\mathfrak{A}} 0) = (1/0)$. This contradiction completes the argument. □

By Theorem 6.120(i)⇒(ii) and Lemma 6.126, we immediately have:

Corollary 6.127. *Suppose $\mathcal{A}$ is both implicative (and so disjunctive) and conjunctive (in particular, negative; cf. Remark 2.9(i)***a***)) [in particular, both disjunctive and negative; cf. Remark 2.9(i)***c***)]. Then, C is not self-extensional, unless it is $\sim$-classical.*

This immediately both shows that Gödel's three-valued logic [4], though being weakly implicative, is not implicative, and covers three-valued expansions of L_3, LA, HZ and P^1, those of the former being equally covered by:

Corollary 6.128. *Suppose $\mathcal{A}$ is both truth-singular (in particular, both $\underline{\vee}$-disjunctive and $(\underline{\vee},\sim)$-paracomplete) and $\sqsupset$-implicative. Then, C is not self-extensional, unless it is $\sim$-classical.*

Proof. Then, $(a \sqsupset^{\mathfrak{A}} a) = 1$, for all $a \in A$, in which case $\mathcal{A}$ is $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \sqsupset \sim(x_0 \sqsupset x_0))$, and so Corollary 6.127 completes the argument. □

The "false-singular" case is but more complicated. First, we have:

Corollary 6.129. *Suppose $\mathcal{A}$ is both false-singular and $\sqsupset$-implicative. Then, C is not self-extensional, unless it is either $\sim$-paraconsistent or $\sim$-classical.*

Proof. If C is not $\sim$-paraconsistent, then $\sim^{\mathfrak{A}}\frac{1}{2} = 0$, in which case $\mathcal{A}$ is $\sim$-negative, and so Corollary 6.127 completes the argument. □

Theorem 6.130. *Suppose $\mathcal{A}$ is both $\sqsupset$-implicative (viz., C is so; cf. Lemma 6.31), hereditarily simple (i.e., C is not $\sim$-classical; cf. Corollary 6.36) and false-singular (in particular, $\sim$-paraconsistent [i.e., C is so]). Then, the following are equivalent:*

(i) *C is self-extensional;*
(ii) *$\mathcal{A}_{\frac{1}{2}} \in \mathrm{Mod}(C)$;*
(iii) *$\sim^{\mathfrak{A}}$ is a bijective endomorphism of $\mathfrak{A}$;*
(iv) *h_{1-} is an endomorphism of $\mathfrak{A}$;*
(v) *$\mathcal{A}_{+0}$ is isomorphic to $\mathcal{A}$;*
(vi) *C is defined by $\mathcal{A}_{0+}$;*
(vii) *$\mathcal{A}_{0+} \in \mathrm{Mod}(C)$;*
(viii) *$\mathfrak{A}$ is an $\sqsupset$-implicative inner semilattice having a non-singular non-diagonal {partial} endomorphism.*

Proof. First, assume (i) holds. Then, as $\frac{1}{2} \neq 1$, by Theorem 5.10, there is some $h \in \hom(\mathfrak{A}, \mathfrak{A})$ such that $\chi^{\mathcal{A}}(h(\frac{1}{2})) \neq \chi^{\mathcal{A}}(h(1))$. Moreover, by (2.12), $a \triangleq (\frac{1}{2} \sqsupset^{\mathfrak{A}} \frac{1}{2}) \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$. If a was not equal to $\frac{1}{2}$, then it would be equal to 1, and so would be $(b \sqsupset^{\mathfrak{A}} b)$, for any $b \in A$, in view of (2.12) and Lemma 5.7, in which case $\mathcal{A}$ would be $\neg$-negative, where $(\neg x_0) \triangleq (x_0 \sqsupset \sim(x_0 \sqsupset x_0))$, contrary to Corollary 6.127. Therefore, $a = \frac{1}{2}$, in which case $(b \sqsupset^{\mathfrak{A}} b) = \frac{1}{2}$, for any $b \in A$, in view of (2.12) and Lemma 5.7, and so $h(\frac{1}{2}) = (h(\frac{1}{2}) \sqsupset^{\mathfrak{A}} h(\frac{1}{2})) = \frac{1}{2} \in D^{\mathcal{A}}$. Hence, $h(1) \notin D^{\mathcal{A}}$, in which case $h(1) = 0$, and so $h(0) = h(\sim^{\mathfrak{A}}1) = \sim^{\mathfrak{A}}h(1) = \sim^{\mathfrak{A}}0 = 1$. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_{1-}$, and so (iv) holds.

Next, (iv)⇒(v/iii) is by the fact that $h_{1-} : A \to A$ is bijective and (6.12)/"Claim 6.117".

Conversely, assume (iii) holds. Then, $\sim^{\mathfrak{A}}[A/2] = (A/2)$, in which case $\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$, and so $h_{1-} = \sim^{\mathfrak{A}} \in \hom(\mathfrak{A}, \mathfrak{A})$. Thus, (iv) holds.

Further, (v)⇒(vi) is by (2.23), while (vii) is a particular case of (vi), whereas (vii)⇒(ii) is by the fact that $D^{\mathcal{A}_{\frac{1}{2}}} = (D^{\mathcal{A}} \cap D^{\mathcal{A}_{0+}})$. Furthermore, (ii)⇒(i) is by (6.10) and Theorem 5.1(vi)⇒(i) with $\mathsf{S} = \{\mathcal{A}, \mathcal{A}_{\frac{1}{2}}\}$. Thus, we have proved the equivalence of (i-vii).

Finally, (i,iv)⇒(viii) is by Theorem 5.9 and the fact that $h_{1-}(0) = 1 \neq 0$, while $(\mathrm{img}\, h_{1-}) = A$ is not a singleton. Conversely, assume (viii) holds. Then, $\mathfrak{A}$ is an $\sqsupset$-implicative inner semi-lattice, while there are some subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and some non-singular non-diagonal $h \in \hom(\mathfrak{B}, \mathfrak{A})$, in which case $(\mathrm{img}\, h) \neq \varnothing$ is not a singleton, and so is $B = (\mathrm{dom}\, h) \neq \varnothing$. Hence, $2 \subseteq B$, in which case, $a \triangleq (1 \sqsupset^{\mathfrak{A}} 1) \in B$, and so, by (2.3), $h(a) = (h(1) \sqsupset^{\mathfrak{A}} h(1)) = a$. Moreover, by (2.12), $a \in D^{\mathcal{A}} = \{\frac{1}{2}, 1\}$. Therefore, if a was not equal to $\frac{1}{2}$, then it would be equal to 1, in which case we would have $h(1) = 1$, and so would get $h(0) = h(\sim^{\mathfrak{A}}1) = \sim^{\mathfrak{A}}h(1) = \sim^{\mathfrak{A}}1 = 0$, in which case, by the non-diagonality of h, we would have $\frac{1}{2} \in B$ and $h(\frac{1}{2}) = i$, for some $i \in 2$, and so $h = h_i$ would be an endomorphism of $\mathfrak{A}$, contrary to Lemma 6.126. Thus, $B \ni a = \frac{1}{2}$, in which case $B = A$, while $h(\frac{1}{2}) = \frac{1}{2}$, and so, by the non-diagonality of h, there is some $i \in 2$ such that $h(i) \neq i$.

Let us prove, by contradiction, that $h(i) \neq \frac{1}{2}$. For suppose $h(i) = \frac{1}{2}$. In that case, if $h(1-i)$ was not equal to $\frac{1}{2}$, then it would be equal to some $j \in 2$, and so we would have $\frac{1}{2} = h(i) = h(\sim^{\mathfrak{A}}(1-i)) = \sim^{\mathfrak{A}} h(1-i) = \sim^{\mathfrak{A}} j = (1-j) \in 2$. Hence, $h(1-i) = \frac{1}{2}$, in which case, as $(\operatorname{dom} h) = B = A$ and $\{i, 1-i\} = 2 = (A \setminus \{\frac{1}{2}\})$, we get $(\operatorname{img} h) = \{\frac{1}{2}\}$, contrary to the non-singularity of h. Thus, $h(i) \neq \frac{1}{2}$, in which case $h(i) = (1-i)$, and so $h(1-i) = h(\sim^{\mathfrak{A}} i) = \sim^{\mathfrak{A}} h(i) = \sim^{\mathfrak{A}}(1-i) = i$. Thus, $\hom(\mathfrak{A}, \mathfrak{A}) \ni h = h_{1-}$, and so (iv) holds, as required. ☐

First, by Remark 5.13(v),(iii)**a)**, Lemma 5.14, Corollary 6.128 and Theorem 6.130(i)⇔(viii), we immediately have:

Corollary 6.131. *Suppose $\mathcal{A}$ is $\sqsupset$-implicative (viz., C is so; cf. Lemma 6.31) as well as either $\sim$-paraconsistent or both $\veebar$-disjunctive and $(\veebar, \sim)$-paracomplete (in which case C is so [cf. Lemma 6.30], and so is not $\sim$-classical, while $\{x_0, \sim x_0\}$ is a unary unitary equality determinant for $\mathcal{A}$). Then, C is self-extensional iff the following hold:*

(i) *$\mathcal{A}$ has no equational implication;*
(ii) *$\mathfrak{A}$ is an $\sqsupset$-implicative inner semi-lattice.*

Next, as opposed to Corollary 6.124, by Remark 2.9(ii), Corollaries 6.36, 6.128, Lemma 6.33(ii)⇒(i), Theorem 6.130(i)⇒(iv) and Claim 6.117, we have:

Corollary 6.132. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 6.31) and self-extensional. Then, the following are equivalent:*

(i) *$\mathcal{A}_{\frac{1}{2}}$ is $\sim$-paraconsistent;*
(ii) *$\sim^{\mathfrak{A}}\frac{1}{2} = \frac{1}{2}$;*
(iii) *C is not $\sim$-classical;*
(iv) *$\mathcal{A}$ is not $\sim$-negative.*

Further, by Lemma 6.44, Corollaries 6.128, 6.129, Theorems 6.42(i)⇔(viii), 6.130 (i)⇔(ii) and (2.12), we have:

Corollary 6.133. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 6.31). Then, it has a proper $\sim$-paraconsistent extension iff it is self-extensional and not $\sim$-classical.*

Then, by Corollaries 6.29, 6.47 and 6.133, we first get the following rather minor enhancement of Corollary 6.127:

Corollary 6.134. *Any weakly conjunctive implicative three-valued Σ-logic with subclassical negation $\sim$ is not self-extensional, unless it is $\sim$-classical.*

And what is more, as opposed to Corollary 6.122, by Corollaries 6.58 and 6.133, we have:

Corollary 6.135. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 6.31) and self-extensional. Then, it is $\sim$-subclassical iff it is $\sim$-classical.*

Likewise, by (2.12), Theorem 6.45(iii)**a)**⇒**d)** and Corollary 6.133, we get the following enhancement of the latter:

Corollary 6.136. *Suppose C is both $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 6.31), self-extensional and not $\sim$-classical. Then, $C_{\frac{1}{2}}$ is the only proper ($\sim$-para)consistent extension of C.*

Furthermore, as opposed to Corollary 6.123, we get:

Corollary 6.137. *Suppose $\sqsupset \in \Sigma$ and C is $\sqsupset$-implicative (viz., $\mathcal{A}$ is so; cf. Lemma 6.31). Then, C has PWC w.r.t. $\sim$ iff $\mathcal{A}$ is $\sim$-negative. In particular, [providing C is self-extensional] C has PWC w.r.t. $\sim$ if[f] it is $\sim$-classical. Moreover, any three-valued implicative $\sim$-paraconsistent/"both $\veebar$-disjunctive and $(\veebar,\sim)$-paracomplete" Σ-logic with subclassical negation $\sim$ does not have PWC w.r.t. $\sim$.*

Proof. The "if" parts of the both second and third sentences are by Remark 2.9(i)**b)**. The converse ones are proved by contradiction. For suppose C has PWC w.r.t. $\sim$ but $\mathcal{A}$ is not $\sim$-negative (in particular, C is self-extensional but not $\sim$-classical; cf. Corollary 6.132(iv)$\Rightarrow$(iii)). Let $\Sigma' \triangleq \{\sqsupset, \sim\} \subseteq \Sigma$, in which case $\mathcal{A}' \triangleq (\mathcal{A}{\upharpoonright}\Sigma')$ is both three-valued, $\sim$-super-classical, canonical, $\sqsupset$-implicative and non-$\sim$-negative as well as defines the Σ'-fragment C' of C, and so C' is both $\sqsupset$-implicative and, by Remark 2.9(ii), Corollary 6.36 and Lemma 6.33(ii)$\Rightarrow$(i), non-$\sim$-classical, for $\mathcal{A}'$ is non-$\sim$-negative, as well as has PWC w.r.t. $\sim$. In particular, for any $\langle \phi, \psi\rangle \in \equiv^{\omega}_{C'}$ and any $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$, we have both $\sim\phi \equiv^{\omega}_{C'} \sim\psi$, $(\phi \sqsupset \varphi) \equiv^{\omega}_{C'} (\psi \sqsupset \varphi)$ and $(\varphi \sqsupset \phi) \equiv^{\omega}_{C'} (\varphi \sqsupset \psi)$. Therefore, C' is self-extensional. Hence, as (2.12) is a theorem of C', by Corollary 6.128 and Theorem 6.130(i)$\Rightarrow$(ii), for every $a \in A$, $(a \sqsupset^{\mathfrak{A}} a) = \frac{1}{2}$, in which case, by Corollary 6.129, $\sim^{\mathfrak{A}}(a \sqsupset^{\mathfrak{A}} a) = \sim^{\mathfrak{A}}\frac{1}{2} \in D^{\mathcal{A}}$, and so both $x_0 \sqsupset x_0$ and $\sim(x_0 \sqsupset x_0)$ are theorems of C'. Then, we have $(x_0 \sqsupset x_0) \in C'(\varnothing) \subseteq C'(x_0)$, in which case, by PWC w.r.t. $\sim$, we get $\sim x_0 \in C'(\sim(x_0 \sqsupset x_0)) \subseteq C'(\varnothing) \subseteq C'(x_0)$, and so, by (3.11) with $n = 0$ and $m = 1$, $\sim$ is not a subclassical negation for C'. In this way, Corollary 6.29 /"and Lemma 6.30" does/do yield the fourth sentence, completing the argument. □

Finally, existence of a self-extensional implicative $\sim$-paraconsistent three-valued Σ-logic with subclassical negation $\sim$ is due to Corollary 6.29 and:

Example 6.138. Let $\mathcal{A}$ be false-singular, $\Sigma \triangleq \{\supset, \sim\}$ with binary $\supset$, $\sim^{\mathfrak{A}} \triangleq h_{1-}$ and $\supset^{\mathfrak{A}} \triangleq ((\Delta_A \times \{\frac{1}{2}\}) \cup (\pi_1{\upharpoonright}(A^2 \setminus \Delta_A)))$. Then, $\mathcal{A}$ is both $\supset$-implicative and $\sim$-paraconsistent, and so is C. And what is more, $h_{1-} \in \hom(\mathfrak{A}, \mathfrak{A})$, and so, by Theorem 6.130(iv)$\Rightarrow$(i), C is self-extensional. In particular, by Corollaries 6.133 and 6.135, it has a proper $\sim$-paraconsistent extension but no $\sim$-classical one. On the other hand, let $\mathcal{B}$ be any more canonical three-valued $\sim$-super-classical $\supset$-implicative Σ-matrix, the logic of which is self-extensional and not $\sim$-classical, in which case, by Corollary 6.128, $\mathcal{B}$ is false-singular, while, by Corollary 6.132(iii)$\Rightarrow$(ii), $\sim^{\mathfrak{B}}\frac{1}{2} = \frac{1}{2}$, and so $\sim^{\mathfrak{B}} = \sim^{\mathfrak{A}}$. Then, by Theorem 6.130(i)$\Rightarrow$(ii,iv,viii) and (2.12), $\mathfrak{B}$ is an $\sqsupset$-implicative inner semi-lattice, being a $\uplus_{\supset}$-semilattice with zero $\frac{1}{2} = (a \supset^{\mathfrak{B}} a)$, for all $a \in A$, and endomorphism h_{1-}. In particular, by (2.4), $(\frac{1}{2} \supset^{\mathfrak{B}} i) = i$, for all $i \in 2$. Moreover, by the $\supset$-implicativity of $\mathcal{B}$, we have $(1 \supset^{\mathfrak{B}} 0) = 0$, in which case $1 = h_{1-}(0) = h_{1-}(1 \supset^{\mathfrak{B}} 0) = (h_{1-}(1) \supset^{\mathfrak{B}} h_{1-}(0)) = (0 \supset^{\mathfrak{B}} 1)$, and $b \triangleq (1 \supset^{\mathfrak{B}} \frac{1}{2}) \in D^{\mathcal{B}} = \{\frac{1}{2}, 1\}$, in which case, if b was not equal to $\frac{1}{2}$, then it would be equal to 1, in which case we would have $\frac{1}{2} = (1 \uplus^{\mathfrak{B}}_{\supset} \frac{1}{2}) = (b \supset^{\mathfrak{B}} \frac{1}{2}) = b = 1$, and so $b = \frac{1}{2}$. Hence, $\frac{1}{2} = h_{1-}(\frac{1}{2}) = h_{1-}(b) = (h_{1-}(1) \supset^{\mathfrak{B}} h_{1-}(\frac{1}{2})) = (0 \supset^{\mathfrak{B}} \frac{1}{2})$. Thus, $(c \supset^{\mathfrak{B}} d) = (\frac{1}{2}/d)$, for all $c, d \in B$ such that $c = / \neq d$, in which case $\supset^{\mathfrak{B}} = \supset^{\mathfrak{A}}$, and so $\mathcal{B} = \mathcal{A}$. In this way, by Corollary 6.29 and Lemma 6.31, the above C is a unique three-valued $\supset$-implicative self-extensional non-$\sim$-classical (in particular, $\sim$-paraconsistent) Σ-logic with subclassical negation $\sim$. In particular, given any signature $\Sigma' \supseteq \Sigma$, any self-extensional non-$\sim$-classical $\supset$-implicative three-valued Σ'-logic C' with subclassical negation $\sim$, in which case, by Corollaries 6.29, 6.132(iii)$\Leftrightarrow$(iv) and Theorem 6.38, the characteristic matrix of C' is not $\sim$-negative, and so is its Σ-reduct (in particular, this, being three-valued, $\supset$-implicative, $\sim$-super-classical and canonical, is characteristic for the Σ-fragment C'' of C', C'', being self-extensional, is not $\sim$-classical), is an expansion of C. □

This definitely shows that the justice is, at least, in that, when crooks (like Avron and Beziau et al.) plagiarize somebody else's labor (mine, in that case) and rewrite the genuine history of science for their exclusive benefit (in particular, by means of publishing plagiarized work backdating), they inevitably lose the capability (if any was at all ever) of obtaining and publishing new *and correct* results.

7. Conclusions

Aside from quite useful general results and their equally illustrative generic applications (sometimes, even multiple ones providing different insights, and so demonstrating the power of unversal tools elaborated here) to infinite classes of particular logics, the paper demonstrates the value of the conception of equality determinant going back to [21, 22].

Among other things, profound connections between the self-extensionality of unitary finitely-valued logics with unary unitary equality determinant as well as "lattice conjunction and disjunction"/"implicative inner semi-lattice implication" and the algebraizability (in the sense of [18]) of two-side sequent calculi (associated according to [21]) and equivalent (in the sense of [18]) many-place ones (associated according to [23]) /"as well as the logics themselves" discovered here are especially valuable within the context of General Algebraic Logic going back to [15, 18, 19, 22]. In this connection, the "implicative" analogue of Theorem 15 of [22] — Lemma 5.14 — being essentially due to that of Lemma 11 therein — Lemma 5.12 — looks especially remarkable.

Likewise, deep connections between the self-extensionality/"absence of classical extensions"/"structural completeness" of implicative/"non-maximally paraconsistent"/ paraconsistent|"disjunctive paracomplete" three-valued logics with subclassical negation and their ([pre]maximal) paraconsistency|paracompleteness discovered here deserve a particular emphasis within the context of Many-Valued (more generally, Non-Classical) Logic.

Perhaps, a most acute problem remained still open is whether Theorem 6.114 is extendable beyond conjunctive and/or unitary three-valued logics with subclassical negation.

Likewise, within the framework of those $\sim$-paraconsistent/ "implicative $(\veebar, \sim)$-paracomplete" three-valued $\sim$-subclassical Σ-logics with lattice conjunction and disjunction $\veebar$, $(L/K)_5$ forms a subalgebra of the direct square of the underlying algebra of the characteristic matrices of which, the following quite non-trivial universal problems remained open:

(1) What is a relative axiomatization of the logic of $(\mathcal{L}/\mathcal{K})_5$?
(2) What is the lattice of those extensions of $C^{\mathrm{DMP}/(\mathrm{INPC}+\mathrm{DN})}$, $(\mathcal{L}/\mathcal{K})_5$ is a model of which?
(3) What is a class of matrices defining $C^{\mathrm{DMP}/(\mathrm{INPC}+\mathrm{DN})}$?

We conjecture that $C^{\mathrm{DMP}/(\mathrm{INPC}+\mathrm{DN})}$ is defined by $(\mathcal{L}/\mathcal{K})_5$. On the other hand, though being technically quite non-trivial, these problems are not especially acute logically, because they deal with rather extraordinary algebraic stipulations not typical of any already known instances.

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