A Very Brief Note on the Riemann Hypothesis

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Abstract. Robin's criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We also require the properties of *superabundant numbers*, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. In this note, using Robin's inequality on superabundant numbers, we prove that the Riemann Hypothesis is true.

Keywords: Riemann Hypothesis · Robin's inequality · Sum-of-divisors function · Superabundant numbers · Prime numbers.

1 Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer d divides n. Define $f(n)$ as $\frac{\sigma(n)}{n}$.

Proposition 1. *[3, (2.7) pp. 362]. For* $n > 1$

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}.$$

Proposition 2. *[2, Lemma 2.7 pp. 19]. For* $x \geq 2278382$

$$\prod_{q \leq x} \frac{q}{q-1} \leq e^{\gamma} \times \log x \times (1 + \frac{0.2}{\log^3(x)}).$$

Say $\mathsf{Robin}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n,$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm.

Proposition 3. $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann Hypothesis is true [6, Theorem 1 pp. 188].*

It is known that $\mathsf{Robin}(n)$ holds for many classes of natural numbers n.

Proposition 4. $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *such that* $p \leq e^{31.018189471}$*, where* p *is the largest prime divisor of* n *[7, Theorem 4.2 pp. 4].*

Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer [3, pp. 367]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 5. *If* n *is superabundant, then* n *is a Hardy-Ramanujan integer [1, Theorem 1 pp. 450].*

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \ \textit{for} \ \ (m > 1).$$

Proposition 6. *Every colossally abundant number is superabundant [1, pp. 455].*

Proposition 7. *If the Riemann Hypothesis is false, then there are infinitely many colossally abundant numbers* $n > 5040$ *such that* $\mathsf{Robin}(n)$ *fails [6, Proposition pp. 204].*

In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n.

Proposition 8. *[4, Theorem 2 pp. 2].* $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *such that*

$$\nu_2(n) > \left\lceil \frac{1}{\log 2} \times ((\log 2^{19} \times c)^{\frac{1048576}{1048575}} - \log c) \right\rceil,$$

where $c = 2^{-\nu_2(n)} \times n$ *and* $\lceil \ldots \rceil$ *is the ceiling function.*

Proposition 9. *For large enough superabundant number* n

$$p < 2^{\nu_2(n)-1},$$

where p *is the largest prime divisor of* n *[5, Corollary 4.9 pp. 13].*

Putting all together yields the proof of the Riemann Hypothesis.

2 Central Lemma

Lemma 1. *If the Riemann Hypothesis is false, then there are infinitely many superabundant numbers* n *such that* $\mathsf{Robin}(n)$ *fails.*

Proof. This is a direct consequence of Propositions 6 and 7. □

3 Main Insight

Lemma 2. *Let n be a superabundant number. Suppose that* $\mathsf{Robin}(n)$ *fails. Then,*

$$p > \frac{\log n}{e^{\frac{0.2}{\log^2(p)}}},$$

where p is the largest prime divisor of n.

Proof. Let n be a superabundant number. Let the representation of this superabundant number n be the product of the first k consecutive primes $q_1 < \cdots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ as exponents, since n must be a Hardy-Ramanujan integer by Proposition 5. We assume that $q_k > e^{31.018189471}$ by Proposition 4. So,

$$\prod_{q \leq q_k} \frac{q}{q-1} \leq e^{\gamma} \times \log q_k \times (1 + \frac{0.2}{\log^3(q_k)})$$

by Proposition 2. Since $\mathsf{Robin}(n)$ fails, then

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{q \leq q_k} \frac{q}{q-1} \leq e^{\gamma} \times \log q_k \times (1 + \frac{0.2}{\log^3(q_k)})$$

by Proposition 1. Thus,

$$\log\log n < \log q_k + \frac{0.2}{\log^2(q_k)}$$

and therefore, the proof is done after of using the exponentiation. □

4 Main Theorem

Theorem 1. *The Riemann Hypothesis is true.*

Proof. Let n be a large enough superabundant number. We have

$$\log p < (\nu_2(n) - 1) \times \log 2$$

by Proposition 9, where p is the largest prime divisor of n. $\mathsf{Robin}(n)$ holds when

$$\nu_2(n) - 1 > \frac{1}{\log 2} \times ((\log 2^{19} \times c)^{\frac{1048576}{1048575}} - \log c)$$

by Proposition 8 where $c = 2^{-\nu_2(n)} \times n$. We have to show that

$$\log p > (\log 2^{19} \times c)^{\frac{1048576}{1048575}} - \log c.$$

We only need to prove that

$$\log(p \times c) > (\log 2^{19} \times c)^{\frac{1048576}{1048575}}.$$

We apply the logarithm to the both sides,

$$\log\log(p \times c) > \frac{1048576}{1048575} \times \log\log(2^{19} \times c).$$

So,

$$\frac{\log\log(p \times c)}{\log\log(2^{19} \times c)} > \frac{1048576}{1048575}.$$

We can see that

$$\frac{\log\log(p \times c)}{\log\log(2^{19} \times c)} > 1$$

since $p > e^{31.018189471} > 2^{19}$ by Proposition 4. Suppose that $\mathsf{Robin}(n)$ fails. Then,

$$\frac{\log\log(p \times c)}{\log\log(2^{19} \times c)} > \frac{\log\log\left(\frac{\log n}{e^{\frac{0.2}{\log^2(p)}}} \times c\right)}{\log\log(2^{19} \times c)}$$

by Lemma 2. Hence, it is enough to show that

$$\frac{\log\log\left(\frac{\log n}{e^{\frac{0.2}{\log^2(p)}}} \times c\right)}{\log\log(2^{19} \times c)} \geq \frac{1048576}{1048575}.$$

Note that, the left hand side is always increasing and the right hand side is simply a small constant $\frac{1048576}{1048575}$. Consequently, for a large enough superabundant number n, the previous inequality is always satisfied and so, $\mathsf{Robin}(n)$ holds by Proposition 8. We obtain a contradiction under the assumption that $\mathsf{Robin}(n)$ fails. Finally, the study of this arbitrarily selected large enough superabundant number n has revealed that $\mathsf{Robin}(n)$ holds on anyway. Accordingly, $\mathsf{Robin}(n)$ holds for all large enough superabundant numbers n. This contradicts the fact that there are infinite superabundant numbers n, such that $\mathsf{Robin}(n)$ fails when the Riemann Hypothesis is false according to Lemma 1. By reductio ad absurdum, we prove that the Riemann Hypothesis is true. □

5 Conclusions

Practical uses of the Riemann Hypothesis include many propositions that are known to be true under the Riemann Hypothesis, and some that can be shown to be equivalent to the Riemann Hypothesis. Indeed, the Riemann Hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf Hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann Hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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