Another Criterion for the Riemann Hypothesis

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Abstract

Let's define $\delta(x) = (\sum_{q \leq x} \frac{1}{q} - \log \log x - B)$, where $B \approx 0.2614972128$ is the Meissel-Mertens constant. The Robin theorem states that $\delta(x)$ changes sign infinitely often. Let's also define $S(x) = \theta(x) - x$, where $\theta(x)$ is the Chebyshev function. It is known that $S(x)$ changes sign infinitely often. We define the another function $\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B\right)$. We prove that when the inequality $\varpi(x) \leq 0$ is satisfied for some number $x \geq 3$, then the Riemann hypothesis should be false. The Riemann hypothesis is also false when the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied for some number $x \geq 3$ or when $\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log \theta(x)} \leq 1$ is satisfied for some number $x \geq 13.1$ or when there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality $\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log \log \log x)} \leq 1$ is always satisfied for some positive constant C independent of x.

Keywords: Riemann hypothesis, Nicolas inequality, Chebyshev function, prime numbers
2000 MSC: 11M26, 11A41, 11A25

1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [1]. Let $N_n = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times p_n$ denotes a primorial number of order n such that p_n is the n^{th} prime number. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q \mid N_n} \frac{q}{q-1} > e^{\gamma} \times \log \log N_n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, log is the natural logarithm, and $q \mid N_n$ means the prime number q divides to N_n. The importance of this property is:

Theorem 1.1. *[2].* $\mathsf{Nicolas}(p_n)$ *holds for all prime numbers* $p_n > 2$ *if and only if the Riemann hypothesis is true.*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

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where $p \leq x$ means all the prime numbers p that are less than or equal to x. We know these properties for this function:

Theorem 1.2. *[3].*
$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1.$$

Theorem 1.3. *[4]. There are infinitely many values of x such that*

$$\theta(x) > x + C \times \sqrt{x} \times \log\log\log x$$

for some positive constant C independent of x.

Let's define $S(x) = \theta(x) - x$. It is a known result that:

Theorem 1.4. *[5]. $S(x)$ changes sign infinitely often.*

We also know that

Theorem 1.5. *[6]. If the Riemann hypothesis holds, then*

$$\left(\frac{e^{-\gamma}}{\log x} \times \prod_{q\leq x} \frac{q}{q-1} - 1\right) < \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}$$

for all numbers $x \geq 13.1$.

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [7]. We know from the constant H, the following formula:

Theorem 1.6. *[8].*
$$\sum_{q} \left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows

$$u(x) = \sum_{q>x} \left(\log(\frac{q}{q-1}) - \frac{1}{q}\right).$$

Nicolas showed that

Theorem 1.7. *[2]. For $x \geq 2$:*
$$0 < u(x) \leq \frac{1}{2 \times (x-1)}.$$

Let's define:
$$\delta(x) = \left(\sum_{q\leq x} \frac{1}{q} - \log\log x - B\right).$$

Robin theorem states the following result:

Theorem 1.8. *[9]. $\delta(x)$ changes sign infinitely often.*

In addition, the Mertens second theorem states that:

Theorem 1.9. *[7].*

$$\lim_{x\to\infty} \delta(x) = 0.$$

Besides, we use the following theorems:

Theorem 1.10. *[10]. For $x > -1$:*

$$\frac{x}{x+1} \leq \log(1+x) \leq x.$$

Theorem 1.11. *[11]. For $x \geq 1$:*

$$\log(1+\frac{1}{x}) < \frac{1}{x+0.4}.$$

We define another function:

$$\varpi(x) = \left(\sum_{q\leq x} \frac{1}{q} - \log\log\theta(x) - B\right).$$

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $p \leq x$. In this way, we introduce another criterion for the Riemann hypothesis based on the Nicolas criterion and deduce some of its consequences.

2. Results

Theorem 2.1. *The inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $p \leq x$.*

Proof. We start from the inequality:

$$\varpi(x) > u(x)$$

which is equivalent to

$$\left(\sum_{q\leq x} \frac{1}{q} - \log\log\theta(x) - B\right) > \sum_{q>x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right).$$

Let's add the following formula to the both sides of the inequality,

$$\sum_{q\leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right)$$

and due to the theorem 1.6, we obtain that

$$\sum_{q\leq x} \log(\frac{q}{q-1}) - \log\log\theta(x) - B > H$$

because of

$$H = \sum_{q\leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) + \sum_{q>x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right)$$

and

$$\sum_{q \leq x} \log(\frac{q}{q-1}) = \sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right).$$

Let's distribute it and remove B from the both sides:

$$\sum_{q \leq x} \log(\frac{q}{q-1}) > \gamma + \log\log\theta(x)$$

since $H = \gamma - B$. If we apply the exponentiation to the both sides of the inequality, then we have that

$$\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \times \log\theta(x)$$

which means that $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $p \leq x$. The same happens in the reverse implication. ☐

Theorem 2.2. *The Riemann hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all numbers $x \geq 3$.*

Proof. This is a direct consequence of theorems 1.1 and 2.1. ☐

Theorem 2.3. *If the inequality $\varpi(x) \leq 0$ is satisfied for some number $x \geq 3$, then the Riemann hypothesis should be false.*

Proof. This is an implication of theorems 1.7, 2.1 and 2.2. ☐

Theorem 2.4. *If the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied for some number $x \geq 3$, then the Riemann hypothesis should be false.*

Proof. If the inequalities $\delta(x) \leq 0$ and $S(x) \geq 0$ are satisfied for some number $x \geq 3$, then we obtain that $\varpi(x) \leq 0$ is also satisfied, which means that the Riemann hypothesis should be false according to the theorem 2.3. ☐

Theorem 2.5.

$$\lim_{x \to \infty} \varpi(x) = 0.$$

Proof. We know that $\lim_{x\to\infty} \varpi(x) = 0$ for the limits $\lim_{x\to\infty} \delta(x) = 0$ and $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$. In this way, this is a consequence from the theorems 1.9 and 1.2. ☐

Theorem 2.6. *If the Riemann hypothesis holds, then*

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log\theta(x)} > 1$$

for all numbers $x \geq 13.1$.

Proof. Under the assumption that the Riemann hypothesis is true, then we would have

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times \log x \times \left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right)$$

after of distributing the terms based on the theorem 1.5 for all numbers $x \geq 13.1$. If we apply the logarithm to the both sides of the previous inequality, then we obtain that

$$\sum_{q \leq x} \log(\frac{q}{q-1}) < \gamma + \log\log x + \log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right).$$

That would be equivalent to

$$\sum_{q \leq x} \frac{1}{q} + \sum_{q \leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) < \gamma + \log\log x + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

where we know that

$$\begin{aligned}\log\left(1 + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x}}\right) &< \frac{1}{\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} + 0.4} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 0.4 \times (3 \times \log x + 5)} \\ &= \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}\end{aligned}$$

according to theorem 1.11 since $\frac{8 \times \pi \times \sqrt{x}}{3 \times \log x + 5} \geq 1$ for all numbers $x \geq 13.1$. We use the theorems 1.6 and 1.7 to show that

$$\sum_{q \leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) = H - u(x)$$

and $\gamma = H + B$. So,

$$H - u(x) < H + B + \log\log x - \sum_{q \leq x} \frac{1}{q} + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is the same as

$$H - u(x) < H - \delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}.$$

We eliminate the value of H and thus,

$$-u(x) < -\delta(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2}$$

which is equal to

$$u(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

We know from the theorem 2.1 that $\varpi(x) > u(x)$ for all numbers $x \geq 13.1$ and therefore,

$$\varpi(x) + \frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} > \delta(x).$$

Hence,

$$\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2} > \log\log\theta(x)-\log\log x.$$

Suppose that $\theta(x)=\epsilon\times x$ for some constant $\epsilon > 1$. Then,

$$\begin{aligned}\log\log\theta(x)-\log\log x &= \log\log(\epsilon\times x)-\log\log x\\ &= \log(\log x+\log\epsilon)-\log\log x\\ &= \log\left(\log x\times(1+\frac{\log\epsilon}{\log x})\right)-\log\log x\\ &= \log\log x+\log(1+\frac{\log\epsilon}{\log x})-\log\log x\\ &= \log(1+\frac{\log\epsilon}{\log x}).\end{aligned}$$

In addition, we know that

$$\log(1+\frac{\log\epsilon}{\log x}) \geq \frac{\log\epsilon}{\log\theta(x)}$$

using the theorem 1.10 since $\frac{\log\epsilon}{\log x} > -1$ when $\epsilon > 1$. Certainly, we will have that

$$\log(1+\frac{\log\epsilon}{\log x}) \geq \frac{\frac{\log\epsilon}{\log x}}{\frac{\log\epsilon}{\log x}+1} = \frac{\log\epsilon}{\log\epsilon+\log x} = \frac{\log\epsilon}{\log\theta(x)}.$$

Thus,

$$\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2} > \frac{\log\epsilon}{\log\theta(x)}.$$

If we add the following value of $\frac{\log x}{\log\theta(x)}$ to the both sides of the inequality, then

$$\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2}+\frac{\log x}{\log\theta(x)} > \frac{\log\epsilon}{\log\theta(x)}+\frac{\log x}{\log\theta(x)} = \frac{\log\epsilon+\log x}{\log\theta(x)} = \frac{\log\theta(x)}{\log\theta(x)} = 1.$$

We know this inequality is satisfied when $0 < \epsilon \leq 1$ since we would obtain that $\frac{\log x}{\log\theta(x)} \geq 1$. Therefore, the proof is done. □

Theorem 2.7. *If the inequality* $\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2}+\frac{\log x}{\log\theta(x)} \leq 1$ *is satisfied for some number* $x \geq 13.1$*, then the Riemann hypothesis should be false.*

Proof. This is a direct consequence of theorem 2.6. □

Theorem 2.8. *If there exists some number* $y \geq 13.1$ *such that for all* $x \geq y$ *the inequality* $\frac{3\times\log x+5}{8\times\pi\times\sqrt{x}+1.2\times\log x+2}+\frac{\log x}{\log(x+C\times\sqrt{x}\times\log\log\log x)} \leq 1$ *is always satisfied for some positive constant C independent of x, then the Riemann hypothesis should be false.*

Proof. From the theorem 1.3, we know that there are infinitely many values of x such that

$$\theta(x) > x+C\times\sqrt{x}\times\log\log\log x$$

for some positive constant C independent of x. That would be equivalent to

$$\log \theta(x) > \log(x + C \times \sqrt{x} \times \log\log\log x)$$

and so,

$$\frac{1}{\log \theta(x)} < \frac{1}{\log(x + C \times \sqrt{x} \times \log\log\log x)}$$

for all numbers $x \geq 13.1$. Hence,

$$\frac{\log x}{\log \theta(x)} < \frac{\log x}{\log(x + C \times \sqrt{x} \times \log\log\log x)}.$$

If the Riemann hypothesis holds, then

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log\log\log x)} > 1$$

for those values of x that complies with

$$\theta(x) > x + C \times \sqrt{x} \times \log\log\log x$$

due to the theorem 2.6. By contraposition, if there exists some number $y \geq 13.1$ such that for all $x \geq y$ the inequality

$$\frac{3 \times \log x + 5}{8 \times \pi \times \sqrt{x} + 1.2 \times \log x + 2} + \frac{\log x}{\log(x + C \times \sqrt{x} \times \log\log\log x)} \leq 1$$

is always satisfied for some positive constant C independent of x, then the Riemann hypothesis should be false, because of there are infinitely many values of x which satisfy the inequality in the theorem 1.3 and comply with $x \geq y$ no matter how big could be y. □

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