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Abstract: This paper introduces a new Offset concept in quadratics for future expansion to any polynomial.
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1 Introduction

The concept of "offset" is used to represent and quantify the phase difference, or synchronicity between two or more finite or infinite integer sequences. Another term, which could be used, would be "shift" to maintain the same analogy that is used in programming language commands like "shift string". Even though, whatever the finite sequence of elements that will represent a real given sequence (finite or infinite), its elements may or may not be shifted from any index as a reference, and yet retain the same properties as the recurrence equation.

The term offset is being used here because OEIS adopted this term.

In these studies, we will focus on the offset in infinite sequences in the form of polynomials.

Notice that any infinite integer polynomial sequence has no start and no end. The only start and end are our finite representations of the sequence. This justifies why the recurrence equation or modular arithmetic work in any segment of the sequence. In these studies, the segment of a sequence is a sequence representation in a finite string. We may have infinite many segments of the same infinite sequence. Also, these representations may be shifted one from another. In cases where there is a shift between different segments, then the offset concept will be applied.

Every finite representation of an infinite polynomial sequence is made by the use of an index. The simplest possible index count is always done using Tally counting. Tally counting is the simplest and most intuitive system to represent the Natural numbers $\{1, 2, 3, 4, 5, 6\ldots\}$ used for counting. We also need to make it clear that the count of the index using Tally counting has a starting point. The starting point in Tally counting is the index Zero.

Thus, we can admit that all finite representations of a sequence have a starting point. To this starting point, we are giving the name "*empty index*" similar to the concepts empty sum and empty product.

The concept of offset is present in all equations $Y[y]$ that produce Integers as a function of variable y (index). The Integer polynomial sequence where we cannot detect offset variation is the type $Y[y] = constant$. But, any expression where $Y[y]$ varies as a function of y, offsetting is possible. This is because in any expression we can substitute the index y by a new index $h = y + f$, or, more comprehensively, substitute y by $h = H[f]$ (a function of offset f) which can be a linear or any other desired function.

A very simple example of offset concept is how we usually define positive Even numbers. We say that $Y[y]$ is a positive Even if $Y[y] = 2y$. Very simple like that. And also, we usually want to make the expressions as simple as possible. We could also say that $Y[y]$ is positive Even if $Y[y] = 2y + 2$ or $Y[y] = 2y - 6$ or $Y[y] = 2y + 1234567890$ or $Y[y] = 2y + 2k + 2t$, etc. Mathematically all are perfectly correct. As we can choose for simplicity, we always choose to say only $Y[y] = 2y$.

What about positive Odd numbers? Would it be $Y[y] = 2y + 1$ or $Y[y] = 2y - 1$? Or would it still be $Y[y] = 2y + 3$ or $Y[y] = 2y - 3$? Which one to choose? Although all of the forms are mathematically correct to express Odd numbers, here too, simplicity is always taken into consideration. We always choose $Y[y] = 2y - 1$ because of the simple form of indexes counting in Tally counting. Thus, we will have positive Odd numbers for Tally indexes $y = 1,2,3,4,5,6,\ldots$ only in the equation $Y[y] = 2y - 1$. In the equation $Y[y] = 2y + 1$, when we count the indexes according to Tally counting, the first odd will be 3 and odd 1 will not have been considered because it has an offset from Tally counting. Thus, we consider $Y[y] = 2y + 1$ as an offset of $Y[y] = 2y - 1$. In this example, only $Y[y] = 2y - 1$ covers all positive Odd numbers generated by the Tally count from the empty index.

This means that if we had to define a single and absolute equation for the Odd numbers, for the sake of simplicity and use of universal Tally counting, we would choose the option $Y[y] = 2y - 1$. This would then be our zero offset equation and all other possible ones would be offset or shifted relative to this reference.

Offset means that the same sequence of integers in the form of a polynomial can be represented by infinite many equations. It is like expressions, fractions, angles, etc. We can write them in the simplest form or any other infinitely many not simplified form.

From these initial ideas, we will expand these concepts to all infinite polynomial sequences. In present study, we will start with the quadratic sequences. In the future study, we will expand to any other function.

1.1 Previous conventions:

Because our tables will show vertical sequences where the indexes will be on vertical and because on vertical, we have Y-axis in the XY-plane, so the sequences integers elements have to appear in X-axis as a function of the Y-axis. Due to that, in all these studies we will represent any polynomial equation as being in the function of y, or just function $Y[y]$, or $x = Y[y]$.

1.2 Notation for Polynomials In these studies

Generically we will denote any polynomial element as being $Y[y]$. When we want to draw the polynomial in the XY-plane we will make x in the function of y. In the cartesian plane (square lattice grid) we can consider $x = Y[y]$. In different grid other than cartesian plane $x \neq Y[y]$.

When we want to distinguish the d^{th}-degree of the polynomial, we will notate $Yd[y]$ or $x = Yd[y]$.

When we want to make a p^{th}-power operation on an d^{th}-degree polynomial, we will notate: $(Yd[y])^p$.

- Constant (polynomial degree 0) will be noted as

$$Y0[y] = c$$

- Linear (polynomial 1st-degree) will be noted as

$$Y1[y] = by + c$$

- Quadratic (polynomial 2nd-degree) will be noted as

$$Y2[y] = ay^2 + by + c$$

- Cubic (polynomial 3rd-degree) will be noted as

$$Y3[y] = a_3y^3 + ay^2 + by + c$$

- Quartic (polynomial 4th-degree) will be noted as

$$Y4[y] = a_4y^4 + a_3y^3 + ay^2 + by + c$$

- Quintic (polynomial 5th-degree) will be noted as

$$Y5[y] = a_5y^5 + a_4y^4 + a_3y^3 + ay^2 + by + c$$

And so on for Sextic, Septic, Octic, Nonic, Decic, etc.

Generic equation of polynomial d^{th}-degree:

$$Yd[y] = a_dy^d + a_{d-1}y^{d-1} + \cdots + a_4y^4 + a_3y^3 + ay^2 + by + c$$

Generically, to be used in any recurrence equation, we will adopt these equalities notation:

$$Yd[-3] = e$$
$$Yd[-2] = f$$
$$Yd[-1] = g = x_1$$
$$Yd[0] = h = x_2$$
$$Yd[1] = i = x_3$$
$$Yd[2] = j$$
$$Yd[3] = k$$

1.3 Notation for index direction in any polynomial sequence (to be used in recurrence equations)

Any polynomial Integer sequence has 2 directions. This is the reason any polynomial has 2 recurrence equations. So, if the direction is given by

$$Yd[y] \equiv (\dots, e, f, g, h, i, j, k, \dots) = \backslash(\dots, k, j, i, h, g, f, e, \dots)\backslash$$

then, the reversal direction will be given by

$$\backslash Yd[y]\backslash \equiv (\dots, k, j, i, h, g, f, e, \dots) = \backslash(\dots, e, f, g, h, i, j, k, \dots)\backslash$$

1.4 Inflection point vs. vertex nomenclature

Because of the definition of the inflection point is in differential calculus *"an inflection point, point of inflection, flex, or inflection (British English: inflexion[citation needed]) is a point on a continuous plane curve at which the curve changes from being concave (concave downward) to convex (concave upward), or vice versa*"; and

Because of the definition of the vertex in geometry as being "*In geometry, a vertex (plural: vertices or vertexes) is a point where two or more curves, lines, or edges meet. As a consequence of this definition, the point where two lines meet to form an angle and the corners of polygons and polyhedra are vertices*";

Because *"In the geometry of planar curves, a vertex is a point of where the first derivative of curvature is zero*";

And like all studies between polynomials, no feature or phenomenon indicates that there is a difference in behavior between quadratic and other polynomial orders, then, there is no reason to differentiate the inflection point phenomena in quadratics from other polynomials. So, there is no reason to have different names.

In these studies, we will refer to this phenomenon in our tables, text and figures as being only inflection point, even in quadratics which usually has the usual vertex name. Moreover, higher degrees of polynomials then quadratics, besides inflection point may have two or more turning points. But, the common phenomenon among all polynomials is the inflection point.

The definition of a single Inflection Point nomenclature in common to all polynomials becomes important when we compare the behavior of the offset at all degrees.

In these studies, the coordinates of an inflection point in XY-plane will be given by x_{ip} and y_{ip}. Also, we will denote an inflection point as being $ip(x_{ip}, y_{ip})$.

2 Infinitely many equations, infinitely many parabolas, only one sequence

See the behavior represented in the 3 quadratic sequences below of the form $Y[y] = ay^2 + by + c$. The first set represents the sequence A165900 Values of Fibonacci polynomial, the second represents the sequence A002378 Oblong numbers, and the third represents A002061 Central polygonal numbers.

A165900 Values of Fibonacci polynomial								
x_ip		-1,3	-1,3	-1,3	-1,3	-1,3	-1,3	-1,3
x_focus		-1	-1	-1	-1	-1	-1	-1
.atus Rectur		1	1	1	1	1	1	1
y_ip		-2,5	-1,5	-0,5	0,5	1,5	2,5	3,5
f		-3	-2	-1	0	1	2	3
a		1	1	1	1	1	1	1
b		5	3	1	-1	-3	-5	-7
c		5	1	-1	-1	1	5	11
	10	155	131	109	89	71	55	41
	9	131	109	89	71	55	41	29
	8	109	89	71	55	41	29	19
	7	89	71	55	41	29	19	11
	6	71	55	41	29	19	11	5
	5	55	41	29	19	11	5	1
	4	41	29	19	11	5	1	-1
	3	29	19	11	5	1	-1	-1
	2	19	11	5	1	-1	-1	1
Y[1]	1	11	5	1	-1	-1	1	5
Y[0]	0	5	1	-1	-1	1	5	11
Y[-1]	-1	1	-1	-1	1	5	11	19
	-2	-1	-1	1	5	11	19	29
	-3	-1	1	5	11	19	29	41
	-4	1	5	11	19	29	41	55
	-5	5	11	19	29	41	55	71
	-6	11	19	29	41	55	71	89
	-7	19	29	41	55	71	89	109
	-8	29	41	55	71	89	109	131
	-9	41	55	71	89	109	131	155
	-10	55	71	89	109	131	155	181

A002378 Oblong numbers								
x_ip		-0,3	-0,3	-0,3	-0,3	-0,3	-0,3	-0,3
x_focus		0	0	0	0	0	0	0
.atus Rectun		1	1	1	1	1	1	1
y_ip		-2,5	-1,5	-0,5	0,5	1,5	2,5	3,5
f		-3	-2	-1	0	1	2	3
a		1	1	1	1	1	1	1
b		5	3	1	-1	-3	-5	-7
c		6	2	0	0	2	6	12
	10	156	132	110	90	72	56	42
	9	132	110	90	72	56	42	30
	8	110	90	72	56	42	30	20
	7	90	72	56	42	30	20	12
	6	72	56	42	30	20	12	6
	5	56	42	30	20	12	6	2
	4	42	30	20	12	6	2	0
	3	30	20	12	6	2	0	0
	2	20	12	6	2	0	0	2
Y[1]	1	12	6	2	0	0	2	6
Y[0]	0	6	2	0	0	2	6	12
Y[-1]	-1	2	0	0	2	6	12	20
	-2	0	0	2	6	12	20	30
	-3	0	2	6	12	20	30	42
	-4	2	6	12	20	30	42	56
	-5	6	12	20	30	42	56	72
	-6	12	20	30	42	56	72	90
	-7	20	30	42	56	72	90	110
	-8	30	42	56	72	90	110	132
	-9	42	56	72	90	110	132	156
	-10	56	72	90	110	132	156	182

A002061 Central polygonal numbers								
x_ip		0,8	0,8	0,8	0,8	0,8	0,8	0,8
x_focus		1	1	1	1	1	1	1
.atus Rectun		1	1	1	1	1	1	1
y_ip		-2,5	-1,5	-0,5	0,5	1,5	2,5	3,5
f		-3	-2	-1	0	1	2	3
a		1	1	1	1	1	1	1
b		5	3	1	-1	-3	-5	-7
c		7	3	1	1	3	7	13
	10	157	133	111	91	73	57	43
	9	133	111	91	73	57	43	31
	8	111	91	73	57	43	31	21
	7	91	73	57	43	31	21	13
	6	73	57	43	31	21	13	7
	5	57	43	31	21	13	7	3
	4	43	31	21	13	7	3	1
	3	31	21	13	7	3	1	1
	2	21	13	7	3	1	1	3
Y[1]	1	13	7	3	1	1	3	7
Y[0]	0	7	3	1	1	3	7	13
Y[-1]	-1	3	1	1	3	7	13	21
	-2	1	1	3	7	13	21	31
	-3	1	3	7	13	21	31	43
	-4	3	7	13	21	31	43	57
	-5	7	13	21	31	43	57	73
	-6	13	21	31	43	57	73	91
	-7	21	31	43	57	73	91	111
	-8	31	43	57	73	91	111	133
	-9	43	57	73	91	111	133	157
	-10	57	73	91	111	133	157	183

Table 1. The sequences A165900 Fibonacci, A002378 Oblong, and A002061 Central polygonal numbers in offset range $-3 \leq f \leq 3$. Note: to follow the XY plane, the index y in the table is growing bottom-up.

Perceive in each set occur a shift (phase/offset) of the sequence as long as we change the starting element from the starting element $Y[0]$ at index $y = 0$.

This means that, in each table, from each quadratic sequence to the next we are changing the y_{ip} (coordinate y of the inflection point) following a staircase function by a unit step. Note that we maintain the same value for x_{ip} (coordinate x of the inflection point).

In any case, for each set, the sequence of the elements will always be kept. There is no risk to occur inversions, missing, or scramble between elements. Always the result of the sequence of the elements is the same. The reason is only because of the proper adjustments in the coefficients (a, b, c) that is the subject of the present study.

3 The behavior of sequences in the XY plane

Following the table sets, as an illustration, see in the figure below how the sequence A002378 Oblong numbers behave in XY plane in different offsets from $-4 \leq f \leq +4$, and its respective equations:

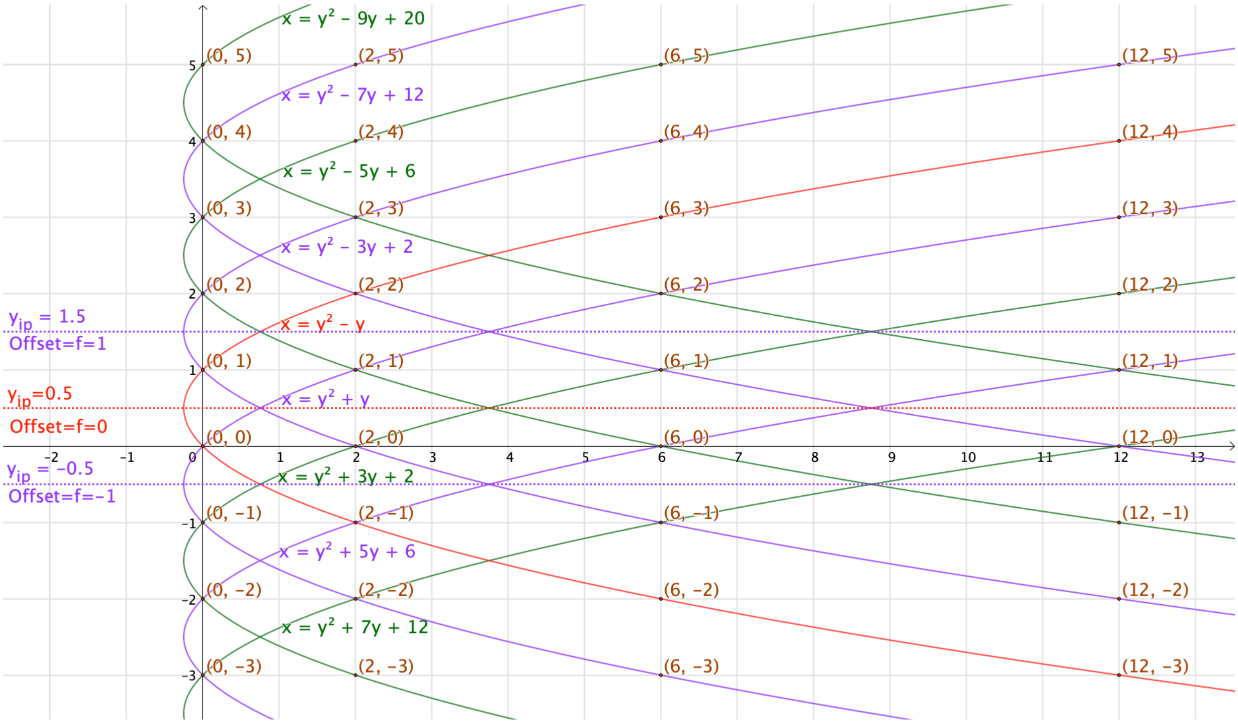

Figure 1. Sequence A002378 Oblong numbers in different offsets in the XY plane

The other two quadratic sequences A165900 Values of Fibonacci polynomial and A002061 Central polygonal numbers are the same as A002378 Oblong numbers, but one shifted one unit to the negative side of the X-axis and the other shifted one unit to the positive side of the X-axis, respectively.

It is clear that in all cases there is no right or wrong choice in which parabola or quadratic equation we have to choose to get the integers sequence desired. For each table set, all parabolas and all equations are correct and generate the same sequence of integers. This happens because as far as we advance the index y, all equations generate the same sequence of elements in the X-axis.

The big difference between them is in the simplicity of the equation. Simplicity in writing and working with. After all, it becomes much easier to work with simplified fractions, small expressions, small numbers, small angles, than large.

4 The core concept of this study

When we have 3 elements $(Y[y_1], Y[y_2], Y[y_3])$ defining completely a quadratic sequence, we mean that each of them is the $Y[y]$ result of the equation $Y[y] = ay^2 + by + c$.
where,

- $Y[y]$ is the function result representing the sequence of the elements,
- (a, b, c) are the fixed coefficients, and
- y is the variable index as an Integer staircase function with constant step 1.

In this case, $Y[y_n]$is the resulting value for each value of y_n. In these studies, y or y_n will be called the index. For each index y, we can calculate the value $Y[y]$ of the element of a quadratic.

Therefore, when we talk about a sequence of 3 elements producing a quadratic, we have defined perfectly the curve of a parabola in the XY plane. We mean that the 3 elements are generated for a sequence of 3 integers values of the index y. In other words, we have a sequence of 3 elements of the quadratic $(Y[y_1], Y[y_2], Y[y_3])$ where each of them was obtained by 3 different indexes (y_1, y_2, y_3):

$$Y[y_1] = ay_1^2 + by_1 + c$$
$$Y[y_1] = ay_2^2 + by_2 + c$$
$$Y[y_1] = ay_3^2 + by_3 + c$$

If $(Y[y_1], Y[y_2], Y[y_3])$ are sequential elements, then y_1, y_2, y_3 are sequential indexes.

Now, to be simpler, we usually take sequential indexes as consecutive integer indexes. So, for consecutive integer indexes y_1, y_2, y_3, we have:

$$y_3 = y_2 + 1 = y_1 + 2$$

Now, considering generically, we have an index y_n to obtain $Y[y_n] = ay_n^2 + by_n + c$:

...
$$for\ y = 3, then\ Y[3] = 9a + 3b + c$$
$$for\ y = 2, then\ Y[2] = 4a + 2b + c$$
$$for\ y = 1, then\ Y[1] = a + b + c$$
$$for\ y = 0, then\ Y[0] = c$$
$$for\ y = -1, then\ Y[-1] = a - b + c$$
$$for\ y = -2, then\ Y[-2] = 4a - 2b + c$$
$$for\ y = -3, then\ Y[-3] = 9a - 3b + c$$
...

Here, perceive that we do not need necessarily 3 consecutive indexes generating 3 consecutive elements to find the 3 coefficients (a, b, c). Any combination of 3 equations will be enough to find the 3 coefficients (a, b, c). But, to be simpler, we will work with 3 consecutive indexes. Likewise, in a straight line: any 2 elements can define it exactly. It will be simpler to use 2 consecutive Integer elements.

Now, the question is: which 3 consecutive indexes should we take from infinite many possibilities to be simpler? Is there a setting that is simplest than all the others? Should it be simpler to use indexes (1,2,3) or indexes (0,1,2) or another set?

5 Analyzing the alternatives

If we consider only the 7 equations above listed as examples and keep the simpler approach using sequential consecutive indexes, we can choose any set of 3 equations from the 5 possibilities below:

- Set 1 of 5:
  - $for\ y_3 = 3, then\ Y[3] = 9a + 3b + c$
  - $for\ y_2 = 2, then\ Y[2] = 4a + 2b + c$
  - $for\ y_1 = 1, then\ Y[1] = a + b + c$
- Set 2 of 5:
  - $for\ y_3 = 2, then\ Y[2] = 4a + 2b + c$
  - $for\ y_2 = 1, then\ Y[1] = a + b + c$
  - $for\ y_1 = 0, then\ Y[0] = c$
- Set 3 of 5:
  - $for\ y_3 = 1, then\ Y[1] = a + b + c$
  - $for\ y_2 = 0, then\ Y[0] = c$
  - $for\ y_1 = -1, then\ Y[-1] = a - b + c$
- Set 4 of 5:
  - $for\ y_3 = 0, then\ Y[0] = c$
  - $for\ y_2 = -1, then\ Y[-1] = a - b + c$
  - $for\ y_1 = -2, then\ Y[-2] = 4a - 2b + c$
- Set 5 of 5:
  - $for\ y_3 = -1, then\ Y[-1] = a - b + c$
  - $for\ y_2 = -2, then\ Y[-2] = 4a - 2b + c$
  - $for\ y_1 = -3, then\ Y[-3] = 9a - 3b + c$

Because of Tally counting, always the first index will be y_1, the second index will be y_2 and the last index will be y_3.

By convention, we'll refer from the first to the last index from the bottom up to remember the direction of the Y-axis in the usual XY plane.

Note that, each set will produce different coefficients (b, c) which will result in different quadratic equations.

Each set of consecutive indexes (y_1, y_2, y_3) will generate 3 different consecutive elements $(Y[y_1], Y[y_2], Y[y_3])$ of the same quadratic integer sequence. But, because all different consecutive elements generate different quadratic equations, then this will be reflected as a different quadratic curve (parabola) in the plane XY. The result will produce the same sequence of elements $Y[y_n]$ shifted one step (one element or one index) from the next. This is the principle of offset.

5.1 Set 1 of 5:

If we look at Robert Sacks' NumberSpiral study, we will conclude it was used this option set 1 of 5:

- $for\ y_3 = 3, then\ k = Y[3] = 9a + 3b + c$
- $for\ y_2 = 2, then\ j = Y[2] = 4a + 2b + c$
- $for\ y_1 = 1, then\ i = Y[1] = a + b + c$

So, when NumberSpiral created 3 simultaneous equations for elements named (i, j, k), they set $y_1 = 1$. And, they wrote: "*the first number,* $Y[1] = i$*, is generated when we plug* $y = 1$ *into* $Y[y] = ay^2 + by + c$ *expression. The next number,*$Y[2] = j$*, since it's the very next number in the sequence after* $Y[1] = i$*, is generated by* $y = 2$*. And similarly,* $Y[3] = k$ *is generated by* $y = 3$."

Then, the result obtained is:

$$a = \frac{i - 2j + k}{2} = \frac{Y[1] - 2Y[2] + Y[3]}{2}$$
$$b = j - i - 3a = Y[2] - Y[1] - 3a$$
$$c = i - a - b = Y[1] - a - b$$

or,

$$a = \frac{i - 2j + k}{2} = \frac{Y[1] - 2Y[2] + Y[3]}{2}$$
$$b = \frac{-5i + 8j - 3k}{2} = \frac{-5Y[1] + 8Y[2] - 3Y[3]}{2}$$
$$c = 3i - 3j + k = 3Y[1] - 3Y[2] + Y[3]$$

Where the final expression of quadratic is given by:

$$Y[y] = \left(\frac{i - 2j + k}{2}\right)y^2 + \left(\frac{-5i + 8j - 3k}{2}\right)y + (3i - 3j + k)$$

or,

$$Y_{Set\ 1\ of\ 5}[y] = \left(\frac{Y[1] - 2Y[2] + Y[3]}{2}\right)y^2 + \left(\frac{-5Y[1] + 8Y[2] - 3Y[3]}{2}\right)y + (3Y[1] - 3Y[2] + Y[3])$$

Perceive that:

- ◊ NumberSpiral used $(y_1 = 1; y_2 = 2; y_3 = 3)$
- ◊ If we set $(y_1 = 2; y_2 = 3; y_3 = 4)$ we get the same coefficient $a = \frac{i - 2j + k}{2} = \frac{Y[1] - 2Y[2] + Y[3]}{2}$, but more complex factors for coefficients b and c than the factors in NumberSpiral set $(y_1 = 1; y_2 = 2; y_3 = 3)$.

So, let's continue in the lower direction and analyze set $(y_1 = 0; y_2 = 1; y_3 = 2)$.

5.2 Set 2 of 5:

Considering,

- $for\ y_3 = 2, then\ Y[2] = 4a + 2b + c$
- $for\ y_2 = 1, then\ Y[1] = a + b + c$
- $for\ y_1 = 0, then\ Y[0] = c$

Then,

$$\begin{aligned} c &= Y[0] \\ Y[1] &= a + b + Y[0] \\ Y[2] &= 4a + 2b + Y[0] \end{aligned}$$

Then,

$$\begin{aligned} 2Y[1] &= 2a + 2b + 2Y[0] \\ Y[2] - 2Y[1] &= 4a + 2b + Y[0] - 2a - 2b - 2Y[0] \\ Y[2] - 2Y[1] &= 2a - Y[0] \\ a &= \frac{Y[0] - 2Y[1] + Y[2]}{2} \end{aligned}$$

Then,

$$\begin{aligned} Y[1] &= a + b + Y[0] \\ Y[1] &= \frac{Y[0] - 2Y[1] + Y[2]}{2} + b + Y[0] \\ 2Y[1] &= Y[0] - 2Y[1] + Y[2] + 2b + 2Y[0] \\ 4Y[1] &= 3Y[0] + Y[2] + 2b \\ b &= \frac{-3Y[0] + 4Y[1] - Y[2]}{2} \end{aligned}$$

Then,

$$Y_{Set\ 2\ of\ 5}[y] = \left(\frac{Y[0] - 2Y[1] + Y[2]}{2}\right) y^2 + \left(\frac{-3Y[0] + 4Y[1] - Y[2]}{2}\right) y + Y[0]$$

5.3 Set 3 of 5:

Considering,

- $for\ y_3 = 1, then\ Y[1] = a + b + c$
- $for\ y_2 = 0, then\ Y[0] = c$
- $for\ y_1 = -1, then\ Y[-1] = a - b + c$

Then,

$$c = Y[0]$$
$$Y[-1] = a - b + Y[0]$$
$$Y[1] = a + b + Y[0]$$

Then,

$$Y[-1] + Y[1] = 2a + Y[0]$$
$$a = \frac{Y[-1] - 2Y[0] + Y[1]}{2}$$

Then,

$$Y[1] - Y[-1] = 2b$$
$$b = \frac{Y[1] - Y[-1]}{2}$$

Then,

$$Y_{Set\ 3\ of\ 5}[y] = \left(\frac{Y[-1] - 2Y[0] + Y[1]}{2}\right)y^2 + \left(\frac{Y[1] - Y[-1]}{2}\right)y + Y[0]$$

5.4 Set 4 of 5:

Considering,

- $for\ y_3 = 0, then\ Y[0] = c$
- $for\ y_2 = -1, then\ Y[-1] = a - b + c$
- $for\ y_1 = -2, then\ Y[-2] = 4a - 2b + c$

Then,

$$c = Y[0]$$
$$Y[-2] = 4a - 2b + Y[0]$$
$$Y[-1] = a - b + Y[0]$$

Then,

$$Y[-2] - 2Y[-1] = 4a - 2b + Y[0] - 2a + 2b - 2Y[0]$$
$$Y[-2] - 2Y[-1] = 2a - Y[0]$$
$$a = \frac{Y[-2] - 2Y[-1] + Y[0]}{2}$$

Then,

$$Y[-2] - 4Y[-1] = 4a - 2b + Y[0] - 4a + 4b - 4Y[0]$$
$$Y[-2] - 4Y[-1] = 2b - 3Y[0]$$
$$b = \frac{x_1 - 4Y[-1] + 3Y[0]}{2}$$

Then,

$$Y_{Set\ 4\ of\ 5}[y] = \left(\frac{Y[-2]\text{-}\ 2Y[-1] + Y[0]}{2}\right)y^2 + \left(\frac{Y[-2] - 4Y[-1] + 3Y[0]}{2}\right)y + Y[0]$$

5.5 Set 5 of 5:

Considering,

- $for\ y_3 = -1, then\ Y[-1] = a - b + c$
- $for\ y_2 = -2, then\ Y[-2] = 4a - 2b + c$
- $for\ y_1 = -3, then\ Y[-3] = 9a - 3b + c$

Then,

$$Y[-2] - Y[-1] = 4a - 2b + c - a + b - c = 3a - b$$
$$b = 3a - Y[-2] + Y[-1]$$

Then,

$$Y[-3] = 9a - 3b + c = 9a - 3(3a - Y[-2] + Y[-1]) + c$$
$$= 3Y[-2] - 3Y[-1] + c$$
$$Y[-2] = 4a - 2b + c = 4a - 2(3a - Y[-2] + Y[-1]) + c$$
$$= 4a - 6a + 2Y[-2] - 2Y[-1] + c = -2a + 2Y[-2] - 2Y[-1] + c$$
$$Y[-3] - Y[-2] = 3Y[-2] - 3Y[-1] + c - (-2a + 2Y[-2] - 2Y[-1] + c)$$
$$= 3Y[-2] - 3Y[-1] + c + 2a - 2Y[-2] + 2Y[-1] - c$$
$$= Y[-2] - Y[-1] + 2a$$
$$a = \frac{Y[-3] - 2Y[-2] + Y[-1]}{2}$$

Then,

$$b = 3a - Y[-2] + Y[-1] = 3\left(\frac{Y[-3] - 2Y[-2] + Y[-1]}{2}\right) - Y[-2] + Y[-1]$$
$$= \frac{3Y[-3] - 6Y[-2] + 3Y[-1] - 2Y[-2] + 2Y[-1]}{2}$$
$$b = \frac{3Y[-3] - 8Y[-2] + 5Y[-1]}{2}$$

Then,

$$c = x_3 - a + b = Y[-1] - \frac{Y[-3] - 2Y[-2] + Y[-1]}{2} + \frac{3Y[-3] - 8Y[-2] + 5Y[-1]}{2}$$
$$c = \frac{2Y[-1] - Y[-3] + 2Y[-2] - Y[-1] + 3Y[-3] - 8Y[-2] + 5Y[-1]}{2}$$
$$c = \frac{2Y[-3] - 6Y[-2] + 6Y[-1]}{2}$$

Then,

$$Y_{Set\ 5\ of\ 5}[y] = \left(\frac{Y[-3]\text{-}\ 2Y[-2] + Y[-1]}{2}\right)y^2 + \left(\frac{3Y[-3] - 8Y[-2] + 5Y[-1]}{2}\right)y + Y[-3] - 3Y[-2] + 3Y[-1]$$

5.6 The conclusion from all sets

Perceive that if we set $(y_1 = -4; y_2 = -3; y_3 = -2)$ we get the same coefficient $a = \frac{Y[1]-2Y[2]+Y[3]}{2}$, but more complex factors for coefficients b and c than the factors in the last set 5 of 5 $(y_1 = -3; y_2 = -2; y_3 = -1)$.

So, let's stop here the analysis.

Then, the 5 sets together can be summarized as:

$$Y_{Set\ 1\ of\ 5}[y] = \left(\frac{Y[1]-2Y[2]+Y[3]}{2}\right)y^2 + \left(\frac{-5Y[1]+8Y[2]-3Y[3]}{2}\right)y + (3Y[1]-3Y[2]+Y[3])$$

$$Y_{Set\ 2\ of\ 5}[y] = \left(\frac{Y[0]-2Y[1]+Y[2]}{2}\right)y^2 + \left(\frac{-3Y[0]+4Y[1]-Y[2]}{2}\right)y + Y[0]$$

$$Y_{Set\ 3\ of\ 5}[y] = \left(\frac{Y[-1]-2Y[0]+Y[1]}{2}\right)y^2 + \left(\frac{Y[1]-Y[-1]}{2}\right)y + Y[0]$$

$$Y_{Set\ 4\ of\ 5}[y] = \left(\frac{Y[-2]-2Y[-1]+Y[0]}{2}\right)y^2 + \left(\frac{Y[-2]-4Y[-1]+3Y[0]}{2}\right)y + Y[0]$$

$$Y_{Set\ 5\ of\ 5}[y] = \left(\frac{Y[-3]-2Y[-2]+Y[-1]}{2}\right)y^2 + \left(\frac{3Y[-3]-8Y[-2]+5Y[-1]}{2}\right)y + (Y[-3]-3Y[-2]+3Y[-1])$$

In conclusion, we do have the simplest general equation for a quadratic function.

The simplest general quadratic equation is

$$Y_{Set\ 3\ of\ 5}[y] = Y[y] = \left(\frac{Y[-1]-2Y[0]+Y[1]}{2}\right)y^2 + \left(\frac{Y[1]-Y[-1]}{2}\right)y + Y[0]$$

6 Quadratic parameters from the simplest equation

Once we have defined our general most simple 2nd-degree polynomial equation as being

$$Y[y] = \left(\frac{Y[-1]-2Y[0]+Y[1]}{2}\right)y^2 + \left(\frac{Y[1]-Y[-1]}{2}\right)y + Y[0]$$

then, let's find all parameters related to it.

To facilitate the visualization of calculations and results, we will make

$$Y[-1] = x_1$$
$$Y[0] = x_2$$
$$Y[1] = x_3$$

This brings us to the re-write the simplest 2nd-degree polynomial equation as

$$Y[y] = \left(\frac{x_1 - 2x_2 + x_3}{2}\right)y^2 + \left(\frac{x_3 - x_1}{2}\right)y + x_2$$

6.1 Coefficient a

$$a = \frac{x_1 - 2x_2 + x_3}{2}$$

6.2 Coefficient b

$$b = \frac{x_3 - x_1}{2}$$

6.3 Coefficient c

$$c = x_2$$

6.4 Equation of the Y coordinate of the inflection point

$$y_{ip} = -\frac{b}{2a} = -\frac{\frac{x_3 - x_1}{2}}{2\frac{x_1 - 2x_2 + x_3}{2}}$$

$$\boxed{y_{ip} = -\frac{b}{2a} = \frac{-(x_3 - x_1)}{2(x_1 - 2x_2 + x_3)}}$$

6.5 Equation of the X coordinate of the inflection point

$$x_{ip} = -\frac{\Delta}{4\text{a}} = -\frac{b^2 - 4ac}{4a} = -\frac{\left(\frac{x_3 - x_1}{2}\right)^2 - 4\left(\frac{x_1 - 2x_2 + x_3}{2}\right)x_2}{4\left(\frac{x_1 - 2x_2 + x_3}{2}\right)}$$

$$= \frac{4\left(\frac{x_1 - 2x_2 + x_3}{2}\right)x_2 - \left(\frac{x_3 - x_1}{2}\right)^2}{2(x_1 - 2x_2 + x_3)} = \frac{2(x_1 - 2x_2 + x_3)x_2 - \frac{(x_3 - x_1)^2}{4}}{2(x_1 - 2x_2 + x_3)}$$

$$= \frac{8(x_1 - 2x_2 + x_3)x_2 - (x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)}$$

$$= \frac{8x_1x_2 - 16x_2^2 + 8x_2x_3 - (x_1^2 + x_3^2 - 2x_1x_3)}{8(x_1 - 2x_2 + x_3)}$$

$$= \frac{8x_1x_2 - 16x_2^2 + 8x_2x_3 - x_1^2 - x_3^2 + 2x_1x_3}{8(x_1 - 2x_2 + x_3)}$$

$$= \frac{-x_1^2 - 16x_2^2 - x_3^2 + 8x_1x_2 + 8x_2x_3 + 2x_1x_3}{8(x_1 - 2x_2 + x_3)}$$

$$= \frac{-x_1^2 - (4x_2)^2 - x_3^2 + 2x_1(4x_2) + 2(4x_2)x_3 + 2x_1x_3}{8(x_1 - 2x_2 + x_3)}$$

Or a more simplified way:

$$x_{ip} = -\frac{b^2 - 4ac}{4a} = c - \frac{b^2}{4a} = x_2 - \frac{\left(\frac{x_3 - x_1}{2}\right)^2}{4\left(\frac{x_1 - 2x_2 + x_3}{2}\right)} = x_2 - \frac{\frac{(x_3 - x_1)^2}{4}}{2(x_1 - 2x_2 + x_3)}$$

$$\boxed{x_{ip} = -\frac{\Delta}{4\text{a}} = -\frac{b^2 - 4ac}{4a} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)}}$$

6.6 Discriminant equation

Note that

$$x_{ip} = -\frac{b^2 - 4ac}{4a} = -\frac{\Delta}{4a}$$

And,

$$x_{ip} = \frac{-x_1^2 - 16x_2^2 - x_3^2 + 8x_1x_2 + 8x_2x_3 + 2x_1x_3}{8(x_1 - 2x_2 + x_3)}$$

$$= -\frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{\frac{16(x_1 - 2x_2 + x_3)}{2}}$$

$$= -\frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{16a}$$

$$x_{ip} = -\frac{\left(\frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4}\right)}{4a}$$

So,

$$\Delta = \frac{x_1^2 + 16x_2^2 + x_3^2 - 8x_1x_2 - 8x_2x_3 - 2x_1x_3}{4}$$

$$\boxed{\Delta = \frac{x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2(4x_2)x_3 - 2x_1x_3}{4}}$$

We know that,

$$(X - Y - Z)^2 = X^2 + Y^2 + Z^2 - 2XY - 2XZ + 2YZ$$

So,

$$X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = (X - Y - Z)^2 - 4YZ$$

Then,

$$X^2 + Y^2 + Z^2 - 2XY - 2XZ - 2YZ = (X - Y - Z + 2\sqrt{YZ})(X - Y - Z - 2\sqrt{YZ})$$

Now, being $X = x_1, Y = 4x_2, Z = x_3$, then:

$$x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2x_1x_3 - 2(4x_2)x_3 = \left(x_1 - 4x_2 - x_3 + 2\sqrt{4x_2x_3}\right)\left(x_1 - 4x_2 - x_3 - 2\sqrt{4x_2x_3}\right)$$

$$x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2x_1x_3 - 2(4x_2)x_3 = (x_1 - 4x_2 - x_3 + 4\sqrt{x_2x_3})(x_1 - 4x_2 - x_3 - 4\sqrt{x_2x_3})$$

$$x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2x_1x_3 - 2(4x_2)x_3 = \left(x_1 - x_3 - 4\left(x_2 - \sqrt{x_2x_3}\right)\right)\left(x_1 - x_3 - 4\left(x_2 + \sqrt{x_2x_3}\right)\right)$$

$$x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2x_1x_3 - 2(4x_2)x_3 = \left(x_1 - x_3 - 4\sqrt{x_2}\left(\sqrt{x_2} - \sqrt{x_3}\right)\right)\left(x_1 - x_3 - 4\sqrt{x_2}\left(\sqrt{x_2} + \sqrt{x_3}\right)\right)$$

$$\boxed{\Delta = \frac{\left(x_1 - x_3 - 4\sqrt{x_2}\left(\sqrt{x_2} - \sqrt{x_3}\right)\right)\left(x_1 - x_3 - 4\sqrt{x_2}\left(\sqrt{x_2} + \sqrt{x_3}\right)\right)}{4}}$$

7 The offset mechanism in quadratics

When we study the general quadratic equation $x = ay^2 + by + c$ we learn that:

- As the second derivative of the equation is only proportional to the coefficient a, then coefficient a is the only one responsible for opening or closing the "mouth" of the quadratic. No other coefficient will change the “mouth” shape of a quadratic. This is only given by “latus rectum”;

$$Latus\ Rectum = LR = \left|\frac{1}{a}\right|$$

  - Note that here is where a quadratic is born. Whatever the equation of a quadratic, it will always be of the form $x = ay^2$.
  - The form of a quadratic always starts at the inflection point in the dot $(X, Y) = (0,0)$ with the orientation of the opening of its “mouth” determined by coefficient a. This is because the form $x = ay^2$ of a quadratic contains the coefficients b and c zeroed and therefore $x_{ip} = 0$ and $y_{ip} = 0$. No matter the value of the coefficient a, the inflection point of the form $x = ay^2$ of a quadratic never leaves the position $(0,0)$.
- If we want to change the position of the inflection point from $(0,0)$ we have to use the other two coefficients (b, c) properly.
  - Once we have coefficient a determined and constant, knowing that $y_{ip} = -\frac{b}{2a}$, then, the coefficient b is the only responsible for the shift (offset/phase) of the inflection point of the quadratic along the Y-axis. Coefficient c does not change the inflection point along Y-axis.
  - Because $x_{ip} = -\frac{b^2-4ac}{4a} = -\frac{b^2}{4a} + c$ then changes in coefficient b also cause a shift of the inflection point along the X-axis.
  - In other words, to shift the inflection point along the Y-axis we have to vary only the coefficient b. But, when we vary the coefficient b for the inflection point to reach the desired value of y_{ip}, we automatically change the value of x_{ip} proportional to the square of the coefficient b.
- Once the general quadratic equation is $x = (something\ 2nd\ degree) + (\text{constant } c)$, then, the coefficient c is responsible only for the shift of the inflection point along the X-axis.
  - This also can be seen in the equation $x_{ip} = -\frac{b^2}{4a} + c$.
  - So, if we want to cancel the inflection point shift caused by the coefficient b on the X-axis, we can compensate it using properly the coefficient c that only changes shift on the X-axis. So, to maintain the x_{ip} fixed, the compensation to be done in coefficient c has to be proportional to b^2.

See in the figure below the offset mechanism showed for A002378 Oblong numbers:

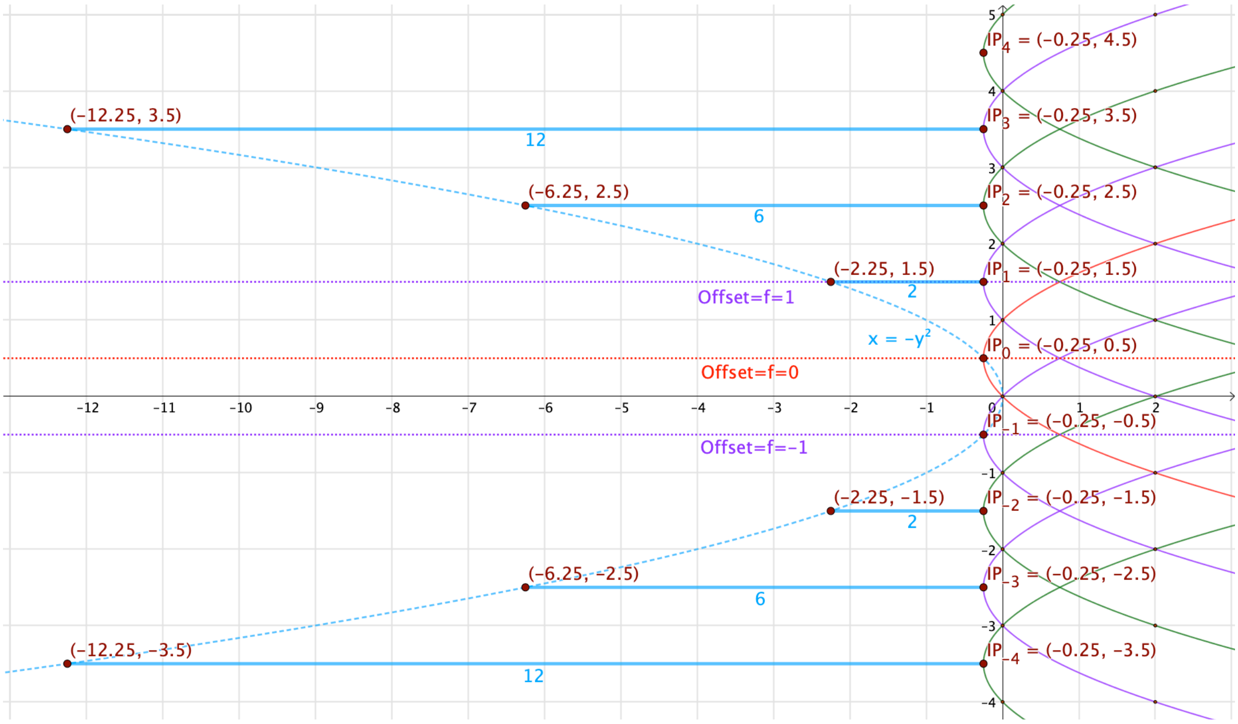

Figure 2. Offset mechanism for sequence A002378 Oblong numbers

Note the distances between the inflection points and the D-parabola $x = -y^2$ follow the sequence of A002378 Oblong numbers $(\ldots, 12, 6, 2, 0, 0, 2, 6, 12, \ldots)$.

7.1 Offset mechanism algebra

Considering the quadratic equation

$$Y[\mathrm{y}] = ay^2 + by + c$$

when we shift the parabola curve along the Y-axis in steps of a unit, we are not changing the possible values that x assumes in the X-axis. For example, when we shift the quadratic $Y[\mathrm{y}] = y^2 + 5y + 6$ in 5 units step up along the Y-axis, then if initially, $Y[-1]$ had the value 2 now $Y[-1+5] = Y[4]$ is also 2.

So, we can express the general quadratic equation above as:

$$Y[y \pm h] = a(y \pm h)^2 + b(y \pm h) + c$$

where

$$x_{ip}[y \pm h] = x_{ip}[y]$$

Whatever the Integer value of h, all equations $Y[y \pm h]$ will always generate the same Integer sequence in $Y[y_n]$. And, we can express:

$$Y[y \pm h] = a(y \pm h)^2 + b(y \pm h) + c$$
$$Y[y \pm h] = a(y^2 \pm 2hy + h^2) + by \pm bh + c$$
$$Y[y \pm h] = ay^2 \pm 2ahy + ah^2 + by \pm bh + c$$
$$\boxed{Y[y \pm h] = ay^2 + (b \pm 2ah)y + (ah^2 \pm bh + c)}$$

7.2 Inflection point X coordinate mechanism with offset

Note that the quadratic curve (parabola) shift along the Y-axis is represented here by variable h in $Y[y \pm h]$. In all cases, we are considering the absolute value of h to express the existing shift.

So, we are predicting:

$Y[y + h] = ay^2 + (b + 2ah)y + (ah^2 + bh + c)$ will shift the parabola in the negative direction along the Y-axis, and

$Y[y - h] = ay^2 + (b - 2ah)y + (ah^2 - bh + c)$ will shift the parabola in the positive direction along the Y-axis.

In both cases, x_{ip} will be the same.

For

$$Y[y] = x = ay^2 + by + c$$

Then,

$$x_{ip}[y] = -\frac{b^2}{4a} + c$$

And, for:

$$Y[y \pm h] = a(y \pm h)^2 + b(y \pm h) + c$$
$$Y[y \pm h] = ay^2 + (b \pm 2ah)y + (ah^2 \pm bh + c)$$
$$x_{ip}[y \pm h] = -\frac{(b \pm 2ah)^2}{4a} + (ah^2 \pm bh + c)$$

When,

$$x_{ip}[y + h] = -\frac{(b + 2ah)^2}{4a} + (ah^2 + bh + c)$$
$$x_{ip}[y + h] = \frac{4a(ah^2 + bh + c)}{4a} - \frac{(b + 2ah)^2}{4a}$$
$$x_{ip}[y + h] = \frac{4a^2h^2 + 4abh + 4ac - b^2 - 4a^2h^2 - 4abh}{4a}$$
$$x_{ip}[y + h] = c - \frac{b^2}{4a} = x_{ip}[\mathrm{y}]$$

Also,

$$x_{ip}[y - h] = -\frac{(b - 2ah)^2}{4a} + (ah^2 - bh + c)$$
$$x_{ip}[y - h] = \frac{4a(ah^2 - bh + c)}{4a} - \frac{(b - 2ah)^2}{4a}$$
$$x_{ip}[y - h] = \frac{4a^2h^2 - 4abh + 4ac - b^2 + 4a^2h^2 - 4abh}{4a}$$
$$x_{ip}[y - h] = -\frac{b^2}{4a} = x_{ip}[\mathrm{y}]$$

7.3 Inflection point Y coordinate mechanism with offset

Considering,

$$Y[\mathrm{y}] = x = ay^2 + by + c$$

$$y_{ip} = -\frac{b}{2a}$$

Or, we can express:

$$y_{ip}[h = 0] = -\frac{b}{2a}$$

$$Y[y \pm h] = a(y \pm h)^2 + b(y \pm h) + c$$

$$Y[y \pm h] = ay^2 + (b \pm 2ah)y + (ah^2 \pm bh + c)$$

$$y_{ip}[h] = -\frac{b \pm 2ah}{2a}$$

$$y_{ip}[h] = -\frac{b}{2a} \mp h$$

$$y_{ip}[h] = y_{ip}(h = 0) \mp h$$

So, there is an inversion of sign between y_{ip} and $offset = h$.

This confirms that for $h > 0$:

- $X[y + h] = ay^2 + (b + 2ah)y + (ah^2 + bh + c)$ will be a shift in the parabola in the negative direction along the Y-axis, and
- $X[y - h] = ay^2 + (b - 2ah)y + (ah^2 - bh + c)$ will be a shift in the parabola in the positive direction along the Y-axis.

When we increase offset in positive values, we decrease the value of y_{ip}.

So, for any $X[y \pm h] = ay^2 + (b \pm 2ah)y + (ah^2 \pm bh + c)$, where$|b \pm 2ah| > |b|$ we can always find a new equation where h=0 and still represents the same sequence.

7.4 Quadratics roots mechanism with Offset

Another way to see the effect of the offset is by analyzing what happens to the roots of quadratics as we vary the offset.

See the equation of the roots of the quadratic equation: $x = ay^2 + by + c$.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x_{1,2} = \frac{-b}{2a} \pm \sqrt{\frac{\Delta}{4a^2}}$$

$$x_{1,2} = \frac{-b}{2a} \pm \sqrt{\frac{\Delta}{4a} * \frac{1}{a}}$$

$$x_{1,2} = y_{ip} \pm \sqrt{-x_{ip} * latus\ rectum}$$

When we vary the offset of a quadratic, coefficient a and x_{ip} are constant. The only variable is y_{ip}. So,

$$x_{1,2} = y_{ip} \pm \text{constant}$$

The roots vary proportionally with y_{ip} and opposite with h.

7.5 Coefficient c mechanism with Offset

The roots are given by

$$x_{1,2} = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x_{1,2} = \frac{-b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{4ac}{4a^2}}$$

$$x_{1,2} = \frac{-b}{2a} \pm \sqrt{\left(-\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

$$x_{1,2} = y_{ip} \pm \sqrt{y_{ip}^2 - \frac{c}{a}}$$

But we know that

$$\sqrt{y_{ip}^2 - \frac{c}{a}} = constant = \sqrt{\frac{\Delta}{4a^2}}$$

Then,

$$y_{ip}^2 - \frac{c}{a} = constant = k = \frac{\Delta}{4a^2}$$

$$y_{ip}^2 - \frac{c}{a} = k$$

$$ay_{ip}^2 - c = ak$$

$$c = ay_{ip}^2 - ak$$

$$c = a(y_{ip}^2 - k)$$

Because a and k are constant, then, coefficient c is only proportional to y_{ip}^2.

Remember that

- When in offset zero at $h = 0$, then $x = Y[y = 0] = a * 0 + b * 0 + c = x_2$. When we move the parabola along Y-axis from this initial position, coefficient c will change proportionally with y_{ip}^2.
- Another way to see, coefficient c is only dependent on the square of y_{ip} position.

8 Definition of offset zero

We will define offset zero and denote it as $f = 0$ when $|y_{ip}|$ is the closest as possible or equal to zero.

We are using the absolute value of y_{ip} because we want to count the indexes y according to Tally counting. We want to use the same reasoning that we did in the Introduction of this study with Odd numbers.

In this definition, we are determining offset $f \in \mathbb{Z}$ to follow $y_{ip} \in \mathbb{R}$.

In XY plane, y_{ip} is a Real continuous function and f is an Integer staircase function with constant step 1.

So, the closest $|y_{ip}|$ to zero will be in the range:

$$-0.5 < y_{ip}[@offset\ zero] \leq 0.5$$

$$-0.5 < \frac{x_1 - x_3}{2x_1 - 4x_2 + 2x_3} \leq 0.5$$

$$-1 < \frac{x_1 - x_3}{x_1 - 2x_2 + x_3} \leq 1$$

$$-x_1 + 2x_2 - x_3 < x_1 - x_3 \leq x_1 - 2x_2 + x_3$$

From

$$x_1 - x_3 \leq x_1 - 2x_2 + x_3$$

$$-2x_3 \leq -2x_2$$

Then,

$$x_3 \geq x_2$$

From

$$-x_1 + 2x_2 - x_3 < x_1 - x_3$$

$$2x_2 < 2x_1$$

Then,

$$x_2 < x_1$$

So, the quadratic condition to result in offset zero is:

$$x_3 \geq x_2 < x_1$$

Or

$$Y[1] \geq Y[0] < Y[-1]$$

So, in the simplest quadratic equation, $x_2 = Y[0] = c$ is the “inflection point integer” because it is the closest integer to x_{ip}. This means that once determined x_2, any $x_1 > x_2$ and $x_3 \geq x_2$ will define the quadratic curve with offset zero.

8.1 Offset f and Inflection point Y coordinate

Now we have defined offset zero as being $-0.5 < y_{ip}[@offset\ zero] \leq 0.5$, then

- For $y_{ip} = -1$, offset is $f = -1$.
- For $y_{ip} = -\frac{3}{4}$, offset is $f = -1$.
- For $y_{ip} = -\frac{1}{2}$, offset is $f = -1$.
- For $y_{ip} = -\frac{1}{4}$, offset is $f = 0$.
- For $y_{ip} = 0$, offset is $f = 0$.
- For $y_{ip} = \frac{1}{4}$, offset is $f = 0$.
- For $y_{ip} = \frac{1}{2}$, offset is $f = 0$.
- For $y_{ip} = \frac{3}{4}$, offset is $f = 1$.
- For $y_{ip} = 1$, offset is $f = 1$.

When y_{ip} increases, the offset f will increase and vice-versa. This means we have to match offset as opposed to the shift h imposed to the index y in the polynomial equation $Y[y]$:

$$f = -h$$

So, when offset increases, the "h" displacement decreases and vice-versa.

Also, because

$$-0.5 < y_{ip}[@offset\ zero] \leq 0.5$$

Being $b^{\text{º}}$ the coefficient b in the quadratic equation with offset zero, then

$$-0.5 < -\frac{b^{\text{º}}}{2a} \leq 0.5$$

$$-1 < -\frac{b^{\text{º}}}{a} \leq 1$$

$$-a < -b^{\text{º}} \leq a$$

In conclusion, the condition to have offset zero is:

$$a > b^{\text{º}} \geq -a$$

9 General quadratic equation including offset

Let's define the quadratic equation in offset zero as being

$$Y_{offset=0}[\mathrm{y}] = Y_{-0.5<y_{ip}\leq 0.5}[\mathrm{y}] = Y^{o}[\mathrm{y}] = ay^2 + b^{o}y + c^{o}$$

Then,

$$y_{ip@offset=0} = -\frac{b^{o}}{2a}$$

$$x_{ip@offset=0} = \frac{-\Delta}{4a} = -\frac{b^{o2} - 4ac^{o}}{4a}$$

So,

$$x_{offset\neq zero} = a(y+h)^2 + b^{o}(y+h) + c^{o}$$

$$x_{offset\neq zero} = a(y^2 + h^2 + 2hy) + b^{o}y + b^{o}h + c^{o}$$

$$x_{offset\neq zero} = ay^2 + ah^2 + 2ahy + b^{o}y + b^{o}h + c^{o}$$

$$x_{offset\neq zero} = ay^2 + (2ahy + b^{o}y) + (ah^2 + b^{o}h + c^{o})$$

Which result in the general offset equation concerning the coefficients of offset zero:

$$x_{offset\neq zero} = ay^2 + (b^{o} + 2ah)y + (ah^2 + b^{o}h + c^{o})$$

When we apply offset, there is no alteration in coefficient a. Now, the new b and c coefficients are:

$$b = b^{o} + 2ah$$

$$c = ah^2 + b^{o}h + c^{o}$$

$$y_{ip@offset\neq 0} = -\frac{b}{2a} = -\frac{b^{o} + 2ah}{2a} = -\frac{b^{o}}{2a} - h$$

$$y_{ip@offset\neq 0} = y_{ip@offset=0} - h$$

$$x_{ip@offset\neq 0} = x_{ip@offset=0}$$

$$-\frac{b^2 - 4ac}{4a} = -\frac{b^{o2} - 4ac^{o}}{4a}$$

$$b^2 - 4ac = b^{o2} - 4ac^{o}$$

$$(2ah + b^{o})^2 - 4a(ah^2 + b^{o}h + c^{o}) = b^{o2} - 4ac^{o}$$

$$4a^2h^2 + b^{o2} + 4ahb^{o} - 4a^2h^2 - 4ab^{o}h - 4ac^{o} = b^{o2} - 4ac^{o}$$

$$4a^2h^2 + b^{o2} + 4ahb^{o} - 4a^2h^2 - 4ab^{o}h - 4ac^{o} = b^{o2} - 4ac^{o}$$

$$0 = 0$$

From $y_{ip@offset\neq 0} = -\frac{b^{o}}{2a} - h = y_{ip@offset\neq 0} - h$ equation, we can see that as far as we increase $h > 0$, the value of $y_{ip@offset\neq 0}$ will decrease from $y_{ip@offset=0}$.

So, if we want the offset parameter to follow y_{ip} direction, we have to change the signal between offset and parameter h:

$$offset = f = -h$$

So, from our general offset equation

$$x = ay^2 + (b^{o} - 2af)y + (af^2 - b^{o}f + c^{o})$$

we will have

$$y_{ip} = -\frac{b^{º} - 2af}{2a}$$
$$y_{ip} = f - \frac{b^{º}}{2a}$$

and

$$x_{ip} = -\frac{\Delta}{4a} = -\frac{(b^{º} - 2af)^2 - 4a(af^2 - b^{º}f + c^{º})}{4a} = \frac{4a(af^2 - b^{º}f + c^{º}) - (b^{º} - 2af)^2}{4a}$$
$$= \frac{4a^2f^2 - 4ab^{º}f + 4ac^{º} - (b^{º2} - 4ab^{º}f + 4a^2f^2)}{4a}$$
$$= \frac{4a^2f^2 - 4ab^{º}f + 4ac^{º} - b^{º2} + 4ab^{º}f - 4a^2f^2}{4a}$$
$$= \frac{4a^2f^2 - 4ab^{º}f + 4ac^{º} - b^{º2} + 4ab^{º}f - 4a^2f^2}{4a} = \frac{4ac^{º} - b^{º2}}{4a}$$
$$x_{ip} = -\frac{b^{º2} - 4ac^{º}}{4a} = c^{º} - \frac{b^{º2}}{4a}$$

From x_{ip} and y_{ip} equations above, we can deduct:

- Once a, $b^{º}$ and $c^{º}$ are fixed values x_{ip} is a fixed value for any offset.
- Only y_{ip} varies in function of offset f.
- Starting from the inflection point, when moving on the quadratic curve along with one of the two possible directions, we will increase or decrease the value of the index y along the Y-axis. In any of the quadratics of our example, we will always arrive exactly at the same values of x. This means that anyone generates the same sequence of integers numbers.

9.1 Another approach

Studying the figure 1. above, we have:

- In the curve $x = y^2 - 7y + 12$ from inflection point given by $(x_{ip} = -0.25;\ y_{ip} = 3.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 - 5y + 6$ from inflection point given by $(x_{ip} = -0.25;\ y_{ip} = 2.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 - 3y + 2$ from inflection point given by $(x_{ip} = -0.25;\ y_{ip} = 1.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 - y$ from inflection point given by $(x_{ip} = -0.25;\ y_{ip} = 0.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 + y$ from inflection point given by $(x_{ip} = -0.25;\ y_{ip} = -0.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 + 3y + 2$ from inflection point given by $(x_{ip} = -0.25;\ y_{ip} = -1.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 + 5y + 6$ from inflection point given by $(x_{ip} = -0.25;\ y_{ip} = -2.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$
- In the curve $x = y^2 + 7y + 12$ from inflection point given by $(x_{ip} = -0.25;\ y_{ip} = -3.5)$ we get the sequence $(0, 2, 6, 12, 20, \dots)$

All of them generate the same integer sequence of numbers as far as we increase or decrease y value from y_{ip} as reference.

Also, note that in these cases, to assure Integer sequences y_{ip} varies following the staircase function with constant step 1, and to assure the same Integer sequence among all parabolas x_{ip} remains constant in all cases.

Notice that when we fix a value of the index $y = y_n$ as the same for all curves, each equation will generate a different element value in X-axis.

That's why, when we want to synchronize the sequence generated by them, we have to consider offset in the equation.

9.2 Offset possibilities

Purposely we have an example with a quadratic integer sequence with 2 simplest equations:

1. $x = y^2 - y$ from inflection point ($x_{ip} = -0.25;\ y_{ip} = 0.5$) we get the sequence $(0, 2, 6, 12, 20, \dots)$, and
2. $x = y^2 + y$ from inflection point ($x_{ip} = -0.25;\ y_{ip} = -0.5$) we get the same sequence $(0, 2, 6, 12, 20, \dots)$.

Which one to choose as being offset zero?

We have to define only one equation as being $offset = f = 0$. How to choose?

Option 1:

If we define $x = y^2 + y$ with $y_{ip} = -0.5$ as being $offset = f = 0$, then $x = y^2 - y$ with $y_{ip} = 0.5$ would have $offset = f = 1$.

Consequently, the whole set would be like this:

- In the curve $x = y^2 - 7y + 12$ we have $y_{ip} = 3.5$ and $f = 4$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $4 \le y < \infty$
- In the curve $x = y^2 - 5y + 6$ we have $y_{ip} = 2.5$ and $f = 3$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $3 \le y < \infty$
- In the curve $x = y^2 - 3y + 2$ we have $y_{ip} = 1.5$ and $f = 2$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $2 \le y < \infty$
- In the curve $x = y^2 - y$ we have $y_{ip} = 0.5$ and $f = 1$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $1 \le y < \infty$
- **In the curve $x = y^2 + y$ we have $y_{ip} = -0.5$ and $f = 0$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $0 \le y < \infty$**
- In the curve $x = y^2 + 3y + 12$ we have $y_{ip} = -1.5$ and $f = -1$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-1 \le y < \infty$
- In the curve $x = y^2 + 5y + 12$ we have $y_{ip} = -2.5$ and $f = -2$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-2 \le y < \infty$
- In the curve $x = y^2 + 7y + 12$ we have $y_{ip} = -3.5$ and $f = -3$. Sequence $(0, 2, 6, 12, 20, \dots)$ will appear in index sequence $-3 \le y < \infty$

In this choice the equation seems to be expressed as:

$$f = ceiling[y_{ip}] = ceiling\left[-\frac{b}{2a}\right] = \left\lceil -\frac{b}{2a} \right\rceil$$

Option 2:

If we define $x = y^2 - y$ with $y_{ip} = 0.5$ as being $offset = f = 0$, then $x = y^2 + y$ with $y_{ip} = -0.5$ would have offset $f = -1$.

Consequently, the whole set would be like this:

- In the curve $x = y^2 - 7y + 12$ we have $y_{ip} = 3.5$ and $f = 3$. Sequence $(0, 2, 6, 12, 20, \ldots)$ will appear in index sequence $4 \leq y < \infty$
- In the curve $x = y^2 - 5y + 6$ we have $y_{ip} = 2.5$ and $f = 2$. Sequence $(0, 2, 6, 12, 20, \ldots)$ will appear in index sequence $3 \leq y < \infty$
- In the curve $x = y^2 - 3y + 2$ we have $y_{ip} = 1.5$ and $f = 1$. Sequence $(0, 2, 6, 12, 20, \ldots)$ will appear in index sequence $2 \leq y < \infty$
- **In the curve $x = y^2 - y$ we have $y_{ip} = 0.5$ and $f = 0$. Sequence $(0, 2, 6, 12, 20, \ldots)$ will appear in index sequence $1 \leq y < \infty$**
- In the curve $x = y^2 + y$ we have $y_{ip} = -0.5$ and $f = -1$. Sequence $(0, 2, 6, 12, 20, \ldots)$ will appear in index sequence $0 \leq y < \infty$
- In the curve $x = y^2 + 3y + 12$ we have $y_{ip} = -1.5$ and $f = -2$. Sequence $(0, 2, 6, 12, 20, \ldots)$ will appear in index sequence $-1 \leq y < \infty$
- In the curve $x = y^2 + 5y + 12$ we have $y_{ip} = -2.5$ and $f = -3$. Sequence $(0, 2, 6, 12, 20, \ldots)$ will appear in index sequence $-2 \leq y < \infty$
- In the curve $x = y^2 + 7y + 12$ we have $y_{ip} = -3.5$ and $f = -4$. Sequence $(0, 2, 6, 12, 20, \ldots)$ will appear in index sequence $-3 \leq y < \infty$

In this choice the equation seems to be expressed as:

$$f = floor[y_{ip}] = floor\left[-\frac{b}{2a}\right] = \left\lfloor -\frac{b}{2a} \right\rfloor$$

When we refer to the offset of a quadratic Integers sequence generated by equations $Y_1[y] = ay^2 + b_1y + c_1$ and $Y_2[y] = ay^2 + b_2y + c_2$, we just mean that there is a displacement of the terms (elements of the sequence) generated on the X-axis in the function of index y. In XY-plane offset between two identical sequences is equivalent to moving the quadratic curve only along the Y-axis in an integer shift (integer y step) without moving the curve along the X-axis.

Remembering the initial example of odd numbers, we want our sequence to appear using Tally counting.

Therefore, the only option where we will have y_{ip} the closest to zero and we can count the indexes y as Tally counting is just this second option $x = y^2 - y$.

So, from our definition: “offset zero is the equation which results in y_{ip} positive most close to $y = 0$” then, we expect to see this behavior in yellow:

	Function 1	Function 2	Function 3	Function Expected
y_{ip}	floor$[y_{ip}]$	ceiling$[y_{ip}]$	round$[y_{ip}]$	roundz$[y_{ip}]$
3,5	3	4	4	3
3,25	3	4	3	3
3	3	3	3	3
2,75	2	3	3	3
2,5	2	3	3	2
2,25	2	3	2	2
2	2	2	2	2
1,75	1	2	2	2
1,5	1	2	2	1
1,25	1	2	1	1
1	1	1	1	1
0,75	0	1	1	1
0,5	0	1	1	0
0,25	0	1	0	0
0	0	0	0	0
-0,25	-1	0	0	0
-0,5	-1	0	0	-1
-0,75	-1	0	-1	-1
-1	-1	-1	-1	-1
-1,25	-2	-1	-1	-1
-1,5	-2	-1	-1	-2
-1,75	-2	-1	-2	-2
-2	-2	-2	-2	-2
-2,25	-3	-2	-2	-2
-2,5	-3	-2	-2	-3
-2,75	-3	-2	-3	-3
-3	-3	-3	-3	-3
-3,25	-4	-3	-3	-3
-3,5	-4	-3	-3	-4

Table 2. Searching the correct equation function for the Offset

9.3 Conclusion

In conclusion, there is no current equation function to express exactly the result of the offset. So, we will introduce a new mathematical function called roundz to calculate the offset.

10 Formal offset equation

The closest equation to the desired result is the round equation. The problem with the round equation appears when $y_{ip} = 0.5$ or any other $y_{ip} = \frac{\text{Odd}}{2}$. For example, the offset for $f_{@(y_{ip}=0.5)} = 0$, but $round[0.5] = 1$.

$$round[y_{ip}] = 0\ for\ (-0.5 \leq y_{ip} < 0.5)$$

So, the round function should be changed to meet offset zero ($f = 0$) for $-0.5 < y_{ip} \leq +0.5$.

$$Offset = f = 0\ when\ -0.5 < y\_ip\ (@offset = 0) = -b/2a \leq +0.5$$

If we create a new equation function called "round to zero" abbreviated as "roundz" where

$$roundz[y_{ip}] = 0\ for\ (-0.5 < y_{ip} \leq\ 0.5)$$

Then, we will meet the desired behavior:

$f = 0$ when $-0.5 < -\frac{b}{2a} \leq +0.5$

Expanding:

$f = -3$ when $-3.5 < -\frac{b}{2a} \leq -2.5$

$f = -2$ when $-2.5 < -\frac{b}{2a} \leq -1.5$

$f = -1$ when $-1.5 < -\frac{b}{2a} \leq -0.5$

$f = 0$ when $-0.5 < -\frac{b}{2a} \leq 0.5$

$f = 1$ when $0.5 < -\frac{b}{2a} \leq 1.5$

$f = 2$ when $1.5 < -\frac{b}{2a} \leq 2.5$

$f = 3$ when $2.5 < -\frac{b}{2a} \leq 3.5$

In the computer, these calculations are almost equivalent to use the function "round" but adjusted to meet the new round to zero (roundz) function.

So, from now on we will define the equation of offset as being:

$$f = roundz\left[-\frac{b}{2a}\right] = round\left[-\frac{b}{2a} - infinitesimal\right]$$

So,

$$offset = f = round\left[-\frac{b}{2a} - \varepsilon\right] = round\left[f - \frac{bº}{2a} - \varepsilon\right]$$

Or using a new function "roundz" (Round Minus Infinitesimal or Round to Zero)

$$offset = f = roundz\left[-\frac{b}{2a}\right] = roundz\left[f - \frac{bº}{2a}\right]$$

11 Example I: offset zero in Fibonacci polynomial

Question: which quadratic equation should we use to represent A165900 in offset zero? Which should be considered the <u>simplest</u> equation $Y[y] = y^2 + y - 1$ or $Y[y] = y^2 - y - 1$?

Let's define:

$$\varphi = \sqrt{1.25} - 0.5 = 0.618033988749895\ldots$$
$$\Phi = \sqrt{1.25} + 0.5 = 1.618033988749895\ldots$$

So,

$$\frac{1}{\varphi} = \frac{1}{\sqrt{1.25} - 0.5} = \frac{\sqrt{1.25} + 0.5}{(\sqrt{1.25} - 0.5)(\sqrt{1.25} + 0.5)} = \frac{\Phi}{1.25 - 0.25} = \Phi$$
$$\frac{1}{\Phi} = \frac{1}{\sqrt{1.25} + 0.5} = \frac{\sqrt{1.25} - 0.5}{(\sqrt{1.25} + 0.5)(\sqrt{1.25} - 0.5)} = \frac{\varphi}{1.25 - 0.25} = \varphi$$
$$\varphi^2 = \left(\sqrt{1.25} - 0.5\right)^2 = 1.25 - \sqrt{1.25} + 0.25 = 1.5 - \sqrt{1.25} = 1 - \varphi$$
$$\Phi^2 = \left(\sqrt{1.25} + 0.5\right)^2 = 1.25 + \sqrt{1.25} + 0.25 = 1.5 + \sqrt{1.25} = 1 + \Phi$$

11.1 Representing A165900 as being $Y[\mathrm{y}] = y^2 + y - 1$

In this case,

$$\varphi^2 + \varphi - 1 = 0$$
$$\varphi^2 + \varphi = 1$$
$$\varphi(\varphi + 1) = 1$$
$$\varphi = \frac{1}{\varphi + 1} = \frac{\varphi - 1}{(\varphi + 1)(\varphi - 1)} = \frac{\varphi - 1}{\varphi^2 - 1} = \frac{\varphi - 1}{1 - \varphi - 1} = \frac{\varphi - 1}{-\varphi} = \frac{1 - \varphi}{\varphi} = \frac{1}{\varphi} - 1$$

Or we could do:

$$\varphi + 1 = \frac{1}{\varphi}$$

Then,

$$\varphi = \frac{1}{\varphi} - 1 = \cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\varphi} - 1} - 1} - 1} - 1} - 1} - 1 = \cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cdots} - 1} - 1} - 1} - 1} - 1} - 1$$

Another approach:

$$\Phi - \varphi = 1$$
$$-\varphi = 1 - \Phi$$

$$-\varphi = 1 - \left(\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\Phi}+1}+1}+1}+1}+1}+1}+1 \right) = \cfrac{-1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\Phi}+1}+1}+1}+1}+1}+1}$$

$$= \cfrac{-1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\varphi+1}+1}+1}+1}+1}+1} = \cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{-\varphi+1}+1}+1}+1}+1}+1}$$

In this option when A165900 is represented by $Y[y] = y^2 + y - 1$, then

- The golden number is a continued fraction less a unit.
- The equation $Y[y] = y^2 + y - 1$ has the sum of the roots equal to -1.
- This is the way how nature decrease.

11.2 Representing A165900 as being $Y[\mathrm{y}] = y^2 - y - 1$

In this case,

$$\Phi^2 - \Phi - 1 = 0$$
$$\Phi^2 - \Phi = 1$$
$$\Phi(\Phi - 1) = 1$$
$$\Phi = \frac{1}{\Phi - 1} = \frac{\Phi + 1}{(\Phi - 1)(\Phi + 1)} = \frac{\Phi + 1}{\Phi^2 - 1} = \frac{\Phi + 1}{1 + \Phi - 1} = \frac{\Phi + 1}{\Phi} = \frac{1}{\Phi} + 1$$

Or we could do:

$$\Phi - 1 = \frac{1}{\Phi}$$

Then,

$$\Phi = \frac{1}{\Phi} + 1 = \cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\Phi}+1}+1}+1}+1}+1} + 1 = \cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cdots}+1}+1}+1}+1}+1} + 1$$

Another approach:

$$\Phi - \varphi = 1$$

$$-\Phi = -1 - \varphi$$

$$-\Phi = -1 - \left(\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\frac{1}{\varphi}-1}-1}-1}-1}-1}-1}-1 \right) = \cfrac{-1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\frac{1}{\varphi}-1}-1}-1}-1}-1}-1}$$

$$= \cfrac{-1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\frac{1}{\Phi-1}-1}-1}-1}-1}-1} = \cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\cfrac{1}{\frac{1}{-\Phi-1}-1}-1}-1}-1}-1}$$

In this option when A165900 is represented by $Y[y] = y^2 - y - 1$, then

- The golden number is a continued fraction plus a unit.
- The equation $Y[y] = y^2 + y - 1$ has the sum of the roots equal to $+1$.
- This is the way how nature grows.

11.3 Conclusion

Because of the way nature grows, A165900 is represented by $Y[y] = y^2 - y - 1$ is the simplest equation with offset zero.

12 Example II: misuse of offset in infinite series

See what happens when we misuse the concept of offset in the infinite series. For example, let's be

$$S = \sum_{n=1}^{\infty} 2^n$$

Then,

$$S = 2 + 4 + 8 + 16 + 32 + 64 + \cdots$$

So,

$$S = 2(1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots)$$

Then,

$$S = 2(1 + S)$$
$$S = 2 + 2S$$
$$S = -2$$

Conclusion:

$$S = \sum_{n=1}^{\infty} 2^n = -2$$

Clearly that a sum of only positive numbers resulting in a negative number indicates that something here is very wrong.

Where is the error?

The error is here

$$S = 2(1 + S)$$

The correct equation should be:

$$S = 2(1 + S_{shifted})$$

Where $S \neq S_{shifted}$.

First that if we use the same concept of continued fraction in the expression $S = 2(1 + S)$ we get

$$S = 2(1 + S)$$
$$S = 2(1 + 2(1 + 2(1 + 2(1 + 2(1 + 2(1 + \cdots))))))$$

This clearly cannot be true for any infinite series S.

We cannot substitute the shifted infinite sum by the original infinite sum again, because they are out of phase, and we are starting from a finite index in the infinite S sum. The correct treatment is:

$$S = \sum_{n=1}^{\infty} 2^n = 2 + 4 + 8 + 16 + 32 + 64 + \cdots$$
$$S = 2\sum_{n=1}^{\infty} 2^{n-1} = 2 + 4 + 8 + 16 + 32 + 64 + \cdots$$
$$S = 2\sum_{n=1}^{\infty} 2^{n-1} = 2(1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots)$$

Or,

$$S = 2\sum_{n=0}^{\infty} 2^n = 2(1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots)$$

So,

$$\frac{S}{2} = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$$

Or,

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots$$

$$\left(\sum_{n=0}^{\infty} 2^n\right) - 1 = 2 + 4 + 8 + 16 + 32 + 64 + \cdots$$

$$\left(\sum_{n=0}^{\infty} 2^n\right) - 1 = S$$

$$\left(\sum_{n=0}^{\infty} 2^n\right) - 1 = \sum_{n=1}^{\infty} 2^n$$

$$\sum_{n=0}^{\infty} 2^n - \sum_{n=1}^{\infty} 2^n = 1$$

At the end we need to operate the equations with indexes synchronized:

$$\sum_{n=1}^{\infty} 2^{n-1} - \sum_{n=1}^{\infty} 2^n = 1$$

$$\sum_{n=1}^{\infty} (2^{n-1} - 2^n) = 1$$

When we start or end an infinite series with a finite index, we do have to respect the offset, the phase, or the synchronism between them. When we respect the offset, we have concise results.

When we are dealing with series covering the infinite indexes without a starting or ending index, then we may write:

$$S = \sum_{y=-\infty}^{\infty} 2^{y+n} = \sum_{y=-\infty}^{\infty} 2^{y+m}$$

Where the integers powers $y + n$ and $y + m$ may or may not be equal. This is because in the infinite the offset does not affect.

In our studies, we interpret the infinite as being the place where the offset does not have meaning.

13 Example III: misuse of offset in infinite series

There is a draft (not published) from Ramanujan where he wrote:

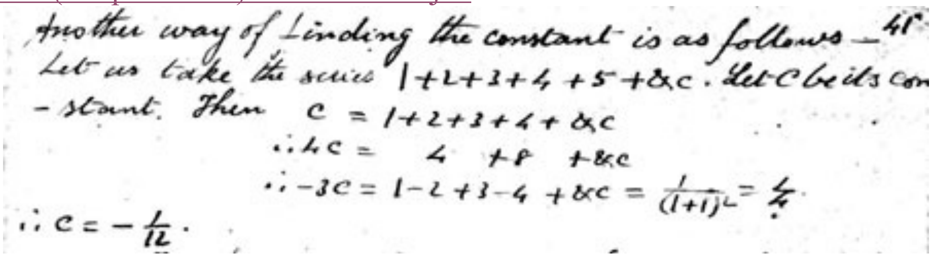
Another way of finding the constant is as follows — 4C
Let us take the series 1+2+3+4+5+&c. Let C be its con-
-stant. Then $C = 1+2+3+4+\&c$
$\therefore 4C = \quad 4 \quad +8 \quad +\&c$
$\therefore -3C = 1-2+3-4+\&c = \frac{1}{(1+1)^2} = \frac{1}{4}$
$\therefore C = -\frac{1}{12}$.

Figure 3. Snippet from Srinivasa Ramanujan's first notebook, chapter 8, concerning an alternate derivation of 1 + 2 + 3 + 4 + ⋯ = −1/12 (Public Domain)

So, he concluded

$$1+2+3+4+5+6+\cdots = -\frac{1}{12}$$

Just because of the signal difference between the two sides of the equality, there is something very confused here.

Similar to the previous example II, Ramanujan wrote

$$c = 1+2+3+4+5+6+\cdots$$
$$4c = 0+4+0+8+0+12+\cdots$$

So, he concluded:

$$c - 4c = 1-2+3-4+5-6+\cdots$$

Where is the error?

The problem here appears when we add zeroes in $4c$. In all finite sums, no problem at all. But, from the perspective of offset in infinite series, this is an error. The reason is that the sum of 2 or more infinite series cannot lose the phase (or synchronism) in the indexes between the series. When we add the zeroes in 4c we are losing the synchronism between the elements from c and 4c.

If

$$c = 1+2+3+4+5+6+\cdots$$

Then,

$$4c = 4+8+12+16+20+24+\cdots$$

And,

$$c - 4c = -3-6-9-12-15-18-\cdots = -3c$$

Because:

$$c = 1+2+3+4+5+6+\cdots = \sum_{y=1}^{\infty} y$$

then,

$$4c = 4+8+12+16+20+24+\cdots = \sum_{y=1}^{\infty} 4y = 4\sum_{y=1}^{\infty} y$$

Now,

$$0+4+0+8+0+12+0+16+0+20+0+24...=\sum_{y=1}^{\infty} 2y\left(\left|cos\left(\frac{y\pi}{2}\right)\right|\right)$$

$$=\sum_{y=1}^{\infty} 2ycos^2\left(\frac{y\pi}{2}\right)$$

So, the error in the Ramanujan draft was to consider the same value the two differences

$$\sum_{y=1}^{\infty} y-4\sum_{y=1}^{\infty} y$$

and

$$=\sum_{y=1}^{\infty} y-\sum_{y=1}^{\infty} 2ycos^2\left(\frac{y\pi}{2}\right)$$

13.1 The Ramanujan differences not considering offset:

$$c-4c=-3c=(1+2+3+4+...)-(4+8+12+16+...)=-(3+6+9+12+...)$$
$$=-3(1+2+3+4+...)=\sum_{y=1}^{\infty} y-\sum_{y=1}^{\infty} 4y=\sum_{y=1}^{\infty}(-3y)=-3\sum_{y=1}^{\infty} y$$

13.2 The Ramanujan differences considering offset:

$$c-\overline{4c}=(1+2+3+4+...)-(0+4+0+8+...)=(1-2+3-4+...)$$
$$=\sum_{y=1}^{\infty} y-\sum_{y=1}^{\infty} 2y\left(\left|cos\left(\frac{y\pi}{2}\right)\right|\right)=\sum_{y=1}^{\infty}\left(y-2y\left(\left|cos\left(\frac{y\pi}{2}\right)\right|\right)\right)$$
$$=\sum_{y=1}^{\infty}\left(y-2ycos^2\left(\frac{y\pi}{2}\right)\right)$$

Note that, the sum $(1-2+3-4+...)$ is the formal power series expansion of the function $\frac{1}{(1-x)^2}=1+2x+3x^2+...+(k+1)x^k+\cdots$ with $x=-1$. Then,

$$(1-2+3-4+...)=\frac{1}{(1+1)^2}=\frac{1}{4}=\sum_{y=1}^{\infty}\left(y-2y\left(\left|cos\left(\frac{y\pi}{2}\right)\right|\right)\right)$$
$$=\sum_{y=1}^{\infty}\left(y-2ycos^2\left(\frac{y\pi}{2}\right)\right)$$

In conclusion:

$$c - 4c = -3c = \sum_{y=1}^{\infty} y - \sum_{y=1}^{\infty} 4y = \sum_{y=1}^{\infty}(-3y) = (-3 - 6 - 9 - 12 - \cdots)$$

and

$$(1 - 2 + 3 - 4 + \ldots) = \sum_{y=1}^{\infty}\left(y - 2y\left(\left|cos\left(\frac{y\pi}{2}\right)\right|\right)\right) = \sum_{y=1}^{\infty}\left(y - 2ycos^2\left(\frac{y\pi}{2}\right)\right) = \frac{1}{4}$$

14 General summary

General 2nd-degree polynomial equation

$$Y[y] = ay^2 + by + c$$

Then, given 3 consecutive elements of the sequence $(Y[y_1], Y[y_2], Y[y_3]) = (x_1, x_2, x_3)$, the simplest equation is

$$Y[y] = \left(\frac{x_1 - 2x_2 + x_3}{2}\right) y^2 + \left(\frac{x_3 - x_1}{2}\right) y + x_2$$

Where the simplest coefficients are:

$$a = \frac{x_1 - 2x_2 + x_3}{2}$$
$$b = \frac{x_3 - x_1}{2}$$
$$c = x_2$$

And

$$y_{ip} = -\frac{b}{2a} = -\frac{x_3 - x_1}{2x_1 - 4x_2 + 2x_3}$$

The offset of this equation is given by:

$$offset = f = roundz[y_{ip}] = roundz\left[-\frac{b}{2a}\right] = roundz\left[-\frac{1}{2} * \frac{x_3 - x_1}{x_1 - 2x_2 + x_3}\right]$$

The discriminant is:

$$\Delta = b^2 - 4ac = \frac{x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2(4x_2)x_3 - 2x_1x_3}{4}$$
$$= \frac{\left(x_1 - x_3 - 4\sqrt{x_2}\left(\sqrt{x_2} - \sqrt{x_3}\right)\right)\left(x_1 - x_3 - 4\sqrt{x_2}\left(\sqrt{x_2} + \sqrt{x_3}\right)\right)}{4}$$

$$x_{ip} = -\frac{\Delta}{4a} = c - \frac{b^2}{4a} = x_2 - \frac{(x_3 - x_1)^2}{8(x_1 - 2x_2 + x_3)}$$
$$= -\frac{x_1^2 + (4x_2)^2 + x_3^2 - 2x_1(4x_2) - 2(4x_2)x_3 - 2x_1x_3}{8(x_1 - 2x_2 + x_3)}$$
$$= -\frac{\left(x_1 - x_3 - 4\sqrt{x_2}\left(\sqrt{x_2} - \sqrt{x_3}\right)\right)\left(x_1 - x_3 - 4\sqrt{x_2}\left(\sqrt{x_2} + \sqrt{x_3}\right)\right)}{8(x_1 - 2x_2 + x_3)}$$

$$Latus\ Rectum = \left|\frac{1}{a}\right| = \left|\frac{2}{x_1 - 2x_2 + x_3}\right|$$

So,

$$Y[y] = ay^2 + by + c = x = ay^2 + (bº - 2af)y + (af^2 - bºf + cº)$$

Then,

$a = aº$	$aº = a$
$b = bº - 2af$	$bº = b + 2af$
$c = aºf^2 - bºf + cº$	$cº = af^2 + bf + c$

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16 References

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