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Frank Vega

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GROUPS PLUS TOURS INC., 9611 Fontainebleau Blvd, Miami, FL, 33172, US

Abstract

The Riemann hypothesis is the assertion that all non-trivial zeros are complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. In 1983, Nicolas stated that the Riemann hypothesis is true if and only if the inequality $\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \cdot \log \theta(x)$ holds for all $x \geq 2$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. In this note, using the Nicolas criterion, we prove that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Riemann zeta function, Prime numbers, Chebyshev function
2000 MSC: 11M26, 11A41, 11A25

1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm. Leonhard Euler studied the following value of the Riemann zeta function (1734).

Proposition 1.1. *It is known that[1, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number. By definition, we have

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

Email address: vega.frank@gmail.com (Frank Vega)

where n denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where π is a well-known irrational number linked to several areas in mathematics such as number theory, geometry, etc.

Proposition 1.2. *For $x \geq 3$ we have [2, Lemma 6.4 pp. 370]:*

$$\left(\prod_{q>x} \frac{q^2}{q^2 - 1} \right) \leq \exp\left(\frac{2}{x}\right),$$

where $\exp(k)$ is the exponential function with value e^k and exponent k. Indeed, Choie and her colleagues proved that for $x \geq 3$ and $t \geq 2$,

$$\log(R_t(x)) \leq \frac{t \cdot x^{1-t}}{t-1},$$

where $R_t(x)$ is given as

$$R_t(x) = \prod_{q>x} (1 - q^{-t})^{-1} = \prod_{q>x} \frac{q^t}{q^t - 1}.$$

Therefore, this Proposition is a particular case of their result applied to the specific value of $t = 2$.

We say that $\mathsf{Nicolas}(x)$ holds provided that

$$\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \cdot \log \theta(x),$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\begin{aligned} \gamma &= \lim_{n\to\infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right) \\ &= \int_1^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor} \right) dx. \end{aligned}$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. Next, we have the Nicolas Theorem:

Proposition 1.3. $\mathsf{Nicolas}(x)$ *holds for all $x \geq 2$ if and only if the Riemann hypothesis is true [3]. Nicolas actually proved that the Riemann hypothesis is true if for all natural numbers $k \geq 1$, then the inequality*

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \cdot \log \log N_k$$

holds, where $N_k = \prod_{i=1}^{k} q_i$ is the primorial number of order k and $\varphi(N_k)$ is the Euler's totient function of N_k [3]. We can state the previous inequality is equivalent to say that $\mathsf{Nicolas}(q_k)$ *holds* *for the kth prime number q_k since we deduce that $\log N_k = \theta(q_k)$. In addition, the well-known Euler's totient function $\varphi(n)$ can be formulated as*

$$\varphi(n) = n \cdot \prod_{q \mid n} \left(1 - \frac{1}{q}\right),$$

where $q \mid n$ means the prime q divides n. Putting all together, we obtain that

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \cdot \log\log N_k$$

holds if and only if

$$\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \cdot \log \theta(x)$$

holds as well, whenever $q_k \leq x$ and there is no other prime different of q_k in the interval $[q_k, x]$. Furthermore, we can use the "if and only if" statement in relation to the Riemann hypothesis since Nicolas also proved that if the Riemann hypothesis is false, then there exist infinitely many natural numbers k for which

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \cdot \log\log N_k$$

does not hold [3].

In number theory, $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function. For $x \geq 2$, a natural number M_x is defined as

$$M_x = \prod_{q \leq x} q.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log\log n}$ for $n \geq 3$.

Proposition 1.4. *Unconditionally on Riemann hypothesis, we know that [4, Proposition 3. pp. 3]:*

$$\lim_{x \to \infty} R(M_x) = \frac{e^{\gamma}}{\zeta(2)}.$$

Indeed, Solé and Planat actually proved that

$$\lim_{k \to \infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

However, we already know that $M_x = N_k$ whenever $q_k \leq x$ and there is no other prime different of q_k in the interval $[q_k, x]$.

The well-known asymptotic notation Ω was introduced by Godfrey Harold Hardy and John Edensor Littlewood [5]. In 1916, they also introduced the two symbols Ω_R and Ω_L defined as [6]:

$$f(x) = \Omega_R(g(x)) \text{ as } x \to \infty \text{ if } \limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0;$$

$$f(x) = \Omega_L(g(x)) \text{ as } x \to \infty \text{ if } \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0.$$

After that, many mathematicians started using these notations in their works. From the last century, these notations Ω_R and Ω_L changed as Ω_+ and Ω_-, respectively. In his seminal paper, Nicolas used another notation: $f(x) = \Omega_\pm(g(x))$ (meaning that $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are both satisfied). Nowadays, the notation $f(x) = \Omega_+(g(x))$ has survived and it is still used in analytic number theory as [7]:

$$f(x) = \Omega_+(g(x)) \text{ if } \exists k > 0\, \forall n_0\, \exists n > n_0 \colon f(n) \geq k \cdot g(n)$$

which has the same meaning to the Hardy and Littlewood older notation. Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

For $x \geq 2$, the function f was introduced by Nicolas in his seminal paper [3]:

$$f(x) = e^\gamma \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

This is a key Lemma.

Lemma 2.1. *If the inequality*

$$\exp\left(\frac{70000000}{\sqrt{x}}\right) \geq f(x)$$

holds for large enough $x \in \mathbb{N}$, then the Riemann hypothesis is true.

Proof. If the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers $x \geq 2$ such that $\log f(x) = \Omega_+(x^{-b})$ [3] (Nicolas actually proved that $\log f(x) = \Omega_\pm(x^{-b})$, but we only need to use the notation Ω_+ in this proof). According to the known definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0) \colon\ \log f(y) \geq k \cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty > 70000000$$

for every possible positive value of k and $b < \frac{1}{2}$. Certainly, no matter how small we can select the absolute value of k, the exponent $-b + \frac{1}{2}$ is always greater than 0 in the expression $y^{-b+\frac{1}{2}} = y^{-b} \cdot \sqrt{y}$. For that reason, we are able to assure that $k \cdot y^{-b} \cdot \sqrt{y}$ goes to infinity whenever y tends to infinity. Thus, there must exist some value of y' such that for all natural numbers $y > y'$ we obtain that the inequality $k \cdot y^{-b} \cdot \sqrt{y} > 70000000$ always holds for an arbitrary value $k > 0$ that we could choose: we pick up the number of 70 million for just simplifying and making a small tribute to the Chinese-American mathematician Yitang Zhang at the same time. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0) \colon\ \log f(y) > \frac{70000000}{\sqrt{y}}.$$

Note that, the variable k disappears in our previous expression due to we do not need it anymore. Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers $x \geq 2$ such that $\log f(x) > \frac{70000000}{\sqrt{x}}$. So, if we have

$$\frac{70000000}{\sqrt{x}} \geq \log f(x)$$

for large enough $x \in \mathbb{N}$, then the Riemann hypothesis cannot be false. In fact, we would obtain that

$$\frac{70000000}{\sqrt{x}} \geq \log f(x) > \frac{70000000}{\sqrt{x}}$$

under the assumption of both conditions. By Reductio ad absurdum, the proof is done after applying the exponentiation to

$$\frac{70000000}{\sqrt{x}} \geq \log f(x)$$

in both sides of the inequality and obtain

$$\exp\left(\frac{70000000}{\sqrt{x}}\right) \geq f(x),$$

since $\frac{70000000}{\sqrt{x}} > \frac{70000000}{\sqrt{x}}$ is a clear contradiction. □

3. Main Theorem

This is the main theorem.

Theorem 3.1. *The Riemann hypothesis is true.*

Proof. If the inequality

$$\exp\left(\frac{70000000}{\sqrt{x}}\right) \geq f(x)$$

holds for large enough $x \in \mathbb{N}$, then the Riemann hypothesis is true by Lemma 2.1. That previous inequality is the same as

$$\exp\left(\frac{70000000}{\sqrt{x}}\right) \cdot \frac{1}{f(x)} \geq 1.$$

We claim that

$$\exp\left(\frac{70000000}{\sqrt{x}}\right) \cdot \frac{1}{f(x)} \geq 1$$

is equivalent to

$$\frac{\exp\left(\frac{70000000}{\sqrt{x}}\right) \cdot \left(\prod_{q \leq x} \frac{q^2}{q^2-1}\right)}{e^{\gamma}} \cdot R(M_x) \geq 1.$$

By definition, we see that

$$
\begin{aligned}
\exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\frac{1}{f(x)} &= \exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\frac{1}{e^{\gamma}\cdot\log\theta(x)\cdot\prod_{q\leq x}\left(1-\frac{1}{q}\right)}\\
&= \frac{\exp\left(\frac{70000000}{\sqrt{x}}\right)}{e^{\gamma}}\cdot\frac{\prod_{q\leq x}\left(\frac{q}{q-1}\right)}{\log\theta(x)}\\
&= \frac{\exp\left(\frac{70000000}{\sqrt{x}}\right)}{e^{\gamma}}\cdot\frac{\prod_{q\leq x}\left(\frac{q+1}{q}\cdot\frac{q^2}{q^2-1}\right)}{\log\theta(x)}\\
&= \frac{\exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right)}{e^{\gamma}}\cdot\frac{\prod_{q\leq x}\left(\frac{q+1}{q}\right)}{\log\theta(x)}\\
&= \frac{\exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right)}{e^{\gamma}}\cdot\frac{M_x\cdot\prod_{q\mid M_x}\left(1+\frac{1}{q}\right)}{M_x\cdot\log\log M_x}\\
&= \frac{\exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right)}{e^{\gamma}}\cdot\frac{\Psi(M_x)}{M_x\cdot\log\log M_x}\\
&= \frac{\exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right)}{e^{\gamma}}\cdot R(M_x)
\end{aligned}
$$

after making some algebra. Moreover, we know that

$$\lim_{x\to\infty} R(M_x) = \frac{e^{\gamma}}{\zeta(2)}$$

by Proposition 1.4. Consequently, there exists a value of x_0 so that for all natural numbers $x \geq x_0$:

$$\liminf_{x\to\infty} R(M_x) - \epsilon = \frac{e^{\gamma}}{\zeta(2)} - \epsilon < R(M_x) < \frac{e^{\gamma}}{\zeta(2)} + \epsilon = \limsup_{x\to\infty} R(M_x) + \epsilon$$

for every arbitrary and absolute value $\epsilon > 0$ (no matter how small we could take the value of $\epsilon > 0$), where by definition of limit superior and inferior we have

$$\liminf_{x\to\infty} R(M_x) = \limsup_{x\to\infty} R(M_x) = \lim_{x\to\infty} R(M_x).$$

On the other hand, the inequality

$$\exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right) \gg \zeta(2)$$

basically holds for large enough $x \in \mathbb{N}$, where $\gg$ means "much greater than" by Propositions 1.1

and 1.2. This is because of

$$\begin{aligned}\exp\left(\frac{70000000}{\sqrt{x}}\right) &\gg \exp\left(\frac{2}{x}\right)\\ &\geq \prod_{q>x} \frac{q^2}{q^2-1}\\ &= \frac{\zeta(2)}{\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right)}\end{aligned}$$

for large enough $x \in \mathbb{N}$, since the inequality

$$\exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right) \gg \zeta(2)$$

is the same as

$$\exp\left(\frac{70000000}{\sqrt{x}}\right) \gg \frac{\zeta(2)}{\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right)}.$$

Since $R(M_x)$ gets closer and closer to $\frac{e^\gamma}{\zeta(2)}$ and simultaneously the inequality

$$\frac{\exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right)}{e^\gamma} \gg \frac{\zeta(2)}{e^\gamma}$$

is more and more evident as long as x increases, then the inequality

$$\frac{\exp\left(\frac{70000000}{\sqrt{x}}\right)\cdot\left(\prod_{q\leq x}\frac{q^2}{q^2-1}\right)}{e^\gamma}\cdot R(M_x) \geq 1$$

necessarily holds for large enough $x \in \mathbb{N}$. In conclusion, we can affirm that the Riemann hypothesis is true because of

$$\exp\left(\frac{70000000}{\sqrt{x}}\right) \geq f(x)$$

feasibly holds for large enough $x \in \mathbb{N}$. □

4. Conclusions

Practical uses of the Riemann hypothesis include many propositions that are known to be true under the Riemann hypothesis and some that can be shown to be equivalent to the Riemann hypothesis. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc. A proof of the Riemann hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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