A Variant of Sharma-Arora's Optimal Eighth-Order Family of Methods for Finding A Simple Root of Nonlinear Equation

Dejan Cebic, Marija Paunovic and Nebojsa Ralevic

May 17, 2018

A Variant of Sharma-Arora's Optimal Eighth-Order Family of Methods for Finding A Simple Root of Nonlinear Equation

Dejan Ćebić
Marija Paunović,
Nebojša M. Ralević

Faculty of Engineering, University of Novi Sad,
Trg Dositeja Obradovića 6,
21000 Novi Sad, Serbia
Email: cebicd@gmail.com, marija.paunovic@uns.ac.rs, nralevic@uns.ac.rs

***Abstract*—In this paper a new variant of Sharma-Arora's family of optimal eighth-order iterative methods for finding simple root of nonlinear equation has considered. The several members of the new modified family are numerically compared with other relevant three-step methods. The numerical performances based on the test examples agree with the theoretical analysis of the presented family.**

Key words. Nonlinear equation, Newton's method, Eighth-order methods, Optimal methods, Efficiency index.

I. Introduction

A vast number of real-life applications require finding a root of a nonlinear equation written in the form $f(x) = 0$. In most cases it is very difficult to find the exact root of nonlinear equation, so many iterative procedures have been created in order to achieve the approximate solution close enough to the exact root. Probably the best-known iterative method is Newton's method (NM) given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \dots \tag{1}$$

For the simple root α (i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$), Newton's method is quadratically convergent for sufficiently good chosen initial point x_0, which means that it generates a sequence $\{x_n\}$ that satisfies

$$\lim_{n\to\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^p} = C, \tag{2}$$

for $p = 2$ and some constant $C \neq 0$.

In 1960. Ostrowski [1] developed the first multistep iterative method introduced with the aim to overcome certain theoretic limitations of the singlestep methods and to increase the convergence order of iterative process. Simultaneously, he proposed the efficiency index suitable for comparison of various iterative methods. Namely, the efficiency index EI is calculated by $EI = p^{1/q}$ where p is the convergence order of the method, while q is the number of function (or derivative) evaluations per iteration. Hence, the efficiency index of Newton's method is $2^{1/2} \approx 1.4142$.

In literature one can find many papers presenting multipoint iterative methods that contain Newton's step (1) (or some of its modifications) as the first step, with greater efficiency indices (see for example [2], [3], [4], [5], [6], [7], [8]). In [9] Kung and Traub conjectured that the iterative method which requires $n + 1$ function evaluations per iteration can reach at most 2^n convergence order in general. The methods that satisfy Kung-Traub conjecture are known as optimal methods (see [10], [11], [12], [13], [14], [15], [16], [14], [18], [19]). This research presents a new family of such optimal eighth order methods.

The further text is organized as follows. The next section shows the Sharma-Arora optimal family and the new similar family of three-step methods. In the third section several well-known iterative methods are employed for the numerical comparison with the new family. Some concluding remarks are given in the last section.

II. A Novel Variant of Sharma-Arora's Family of Methods

Recently, Sharma and Arora [19] have developed an optimal family of three-point eighth-order methods with the following form

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= M_4(x_n, w_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \cdot Q_n, \end{aligned} \tag{3}$$

where

$$Q_n = \frac{f'(x_n) - f[w_n, x_n] + f[z_n, w_n]}{2f[z_n, w_n] - f[z_n, x_n]}, \tag{4}$$

and $f[\cdot, \cdot]$ represents the first order divided difference. It is obvious that the first step is Newton's method, while $M_4(\cdot, \cdot)$ can be any optimal fourth order iterative scheme based on

Newton's method. Thus, method (3) denoted by SA in further text, reaches the eighth convergence order (i.e. satisfies (2) for $p = 8$), requires one derivative and three function evaluations, and therefore, its efficiency index is 1.6818.

The main contribution of this research is the new optimal family of methods with slightly modified body structure compared with SA family (3),

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= M_4(x_n, w_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n) - b \cdot f(z_n)} \cdot Q_n, \end{aligned} \tag{5}$$

whereas the function Q_n is defined by (4), and b is a real parameter. It is obvious that the family (5) reduces to Sharma-Arora's family (3) for $b = 0$. The idea of using the real parameter b has its origins in the research of Wu [20].

The following theorem represents a theoretic analysis of the optimal convergence of the modified family, while its numerical properties including the results for different values of real parameter b are shown in the next section.

Theorem II.1 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable on the open interval D which encloses the simple root α. If x_0 is sufficiently close to α, then the family of methods defined by (5) is of eighth-order.*

Proof: Let $e_n = x_n - \alpha$ denotes the error of the n-th approximation of the method. Let the coefficients c_i be defined by

$$c_i = \frac{1}{i!}\frac{f^{(i)}(\alpha)}{f(\alpha)}$$

for every integer value of $i > 1$. Thus, from the Taylor expansion about α, we have

$$f(x_n) = f'(\alpha)e_n\Big(1 + \sum_{i=2}^{8} c_i e_n^{i-1}\Big) + O(e_n^9) \tag{6}$$

and

$$f'(x_n) = f'(\alpha)\Big(1 + \sum_{i=2}^{8} i c_i e_n^{i-1}\Big) + O(e_n^8). \tag{7}$$

Substitution of (6) and (7) into the Newton's step yields

$$\begin{aligned} w_n = \alpha &- c_2 e_n^2 + \big(-2c_2^2 + 2c_3\big)e_n^3 \\ &+ \big(4c_2^3 - 7c_2c_3 + 3c_4\big)e_n^4 + ... + O(e_n^9), \end{aligned} \tag{8}$$

and consequently

$$\begin{aligned} f(w_n) = f'(\alpha)\big[c_2e_n^2 &- 2\big(c_2^2 - c_3\big)e_n^3 \\ &+ \big(5c_2^3 - 7c_2c_3 + 3c_4\big)e_n^4 + ...\big] + O(e_n^9). \end{aligned} \tag{9}$$

If $M_4(x_n, w_n)$ is some optimal fourth-order iterative scheme, then the error of the second step can be written in the form

$$z_n - \alpha = A_4e_n^4 + A_5e_n^5 + A_6e_n^6 + A_7e_n^7 + A_8e_n^8 + O(e_n^9), \tag{10}$$

for some constants A_i, $i \in \{4, 5, 6, 7, 8\}$. By substituting (10) into (6), it is easy to note that

$$\begin{aligned} f(z_n) = f'(\alpha) \cdot e_n^4 \big(&A_4 + A_5e_n + A_6e_n^2 + A_7e_n^3 \\ &+ (A_8 + A_4^2c_2)e_n^4\big) + O(e_n^9). \end{aligned} \tag{11}$$

By using (11) and (7) one can get

$$\begin{aligned} \tfrac{f(z_n)}{f'(x_n) - bf(z_n)} = A_4e_n^4 &+ \big(A_5 - 2A_4c_2\big)e_n^5 \\ &+ \big(A_6 - 2A_5c_2 + A_4(4c_2^2 - 3c_3)\big)e_n^6 \\ &... + O(e_n^9). \end{aligned} \tag{12}$$

Substitutions of (6), (9) and (11) into the expressions for calculations of divide differences $f[\cdot,\cdot]$ provide a simple form of Taylor's expansion of function Q_n about α, given by

$$\begin{aligned} Q_n = 1 &+ 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 \\ &+ (c_3^2 - c_2c_4 + 5c_5)e_n^4 + O(e_n^5). \end{aligned} \tag{13}$$

Finally, after using (10), (12) and (13) and simplifying, the third iteration step of the family (5) yields

$$x_{n+1} = \alpha - A_4\big(A_4(c_2 + b) + c_3^2 - c_2c_4\big)e_n^8 + O(e_n^9).$$

Therefore, it can be concluded that the family (5) is of optimal eighth order with the error equation

$$e_{n+1} = -A_4\big(A_4(c_2 + b) + c_3^2 - c_2c_4\big)e_n^8 + O(e_n^9).$$

According to the given theorem, the presented modification of SA family satisfies Kung-Traub conjecture of optimality and preserves the efficiency index 1.6818.

Remark: Some expressions such as (8), (9) and (12) are intentionally omitted to display in full form for the sake of simplicity. All the displayed results can be easily verified using Wolfram's Mathematica software for symbolic computations.

III. Numerical Results and Comparison

The members of the family (5) are denoted by **New** and the indices that point out which optimal fourth order method has been taken as the second step of (5). The second step $z_n = M_4(x_n, z_n)$ is chosen as they suggested in Sharma and Arora's research [19]. Namely, method (5) is denoted by New_1, New_2 and New_3, if the second step has a form:

- $z_n = w_n - \dfrac{f(w_n)}{2f[w_n, x_n] - f'(x_n)}$, [1],
- $z_n = w_n - \Big(\dfrac{2}{f[w_n, x_n]} - \dfrac{1}{f'(x_n)}\Big)f(w_n)$, [21],
- $z_n = w_n - \Big(3 - \dfrac{2f[w_n, x_n]}{f'(x_n)}\Big)\dfrac{f(w_n)}{f'(x_n)}$, [19],

respectively. Similarly, when those fourth order schemes are used for Sharma-Arora's family members (3), they are denoted by SA_1, SA_2 and SA_3, respectively. The values of parameter b are chosen such that $|b| = 0.5$ or $|b| = 1.5$.

Other relevant methods chosen for the comparison are well-known three-step iterative schemes with similar theoretical properties as (3) and (5). Namely, every method has efficiency index 1.6818, requires four function/derivative evaluations per iteration, and does not include calculation of second or higher order derivatives.

- Bi, Wu and Ren's method [22] (denoted by BWR):

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= w_n - \frac{2f(x_n) - f(w_n)}{2f(x_n) - 5f(w_n)} \frac{f(w_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{f'(x_n) + (\beta+2)f(z_n)}{f(x_n) + \beta f(z_n)} \\ &\quad \cdot \frac{f(z_n)}{f[z_n, w_n] + f[z_n, x_n, x_n](z_n - w_n)}, \quad \beta \in \mathbb{R}, \end{aligned}$$

where $f[z_n, x_n, x_n] = \frac{f[z_n,x_n] - f'(x_n)}{z_n - x_n}$.

- Thukral and Petković's method [23] (TP):

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= w_n - \frac{f(x_n) + \beta_1 f(w_n)}{f(x_n) + (\beta_1 - 2)f(w_n)} \frac{f(w_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \Big(\phi(t) + \frac{f(z_n)}{f(w_n) - \beta_2 f(z_n)} \\ &\quad + \frac{4f(z_n)}{f(x_n)}\Big) \frac{f(z_n)}{f'(x_n)}, \quad \beta_1, \beta_2 \in \mathbb{R}, \end{aligned}$$

where $\phi(t) = 1 + 2t + (5 - 2\beta_1)t^2 + (12 - 12\beta_1 + 2\beta_1^2)t^3$ and $t = f(w_n)/f(x_n)$.

- Liu and Wang's method [24] (LW):

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= w_n - \frac{f(x_n)}{f(x_n) - 2f(w_n)} \frac{f(w_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \Big[\Big(\frac{f(x_n) - f(w_n)}{f(x_n) - 2f(w_n)}\Big)^2 + \frac{f(z_n)}{f(w_n) - \beta_1 f(z_n)} \\ &\quad + \frac{4f(z_n)}{f(x_n) + \beta_2 f(z_n)}\Big] \frac{f(z_n)}{f'(x_n)}, \quad \beta_1, \beta_2 \in \mathbb{R}. \end{aligned}$$

- Cordero, Torregrosa and Vassileva's method [25] (CTV):

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n) - f(w_n)}{f(x_n) - 2f(w_n)} \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= u_n - S_n \frac{f(z_n)}{f'(x_n)}, \quad \beta_1, \beta_2, \beta_3 \in \mathbb{R}, \end{aligned}$$

where

$$S_n = \frac{3(\beta_2 + \beta_3)(u_n - z_n)}{\beta_1(u_n - z_n) + \beta_2(w_n - x_n) + \beta_3(z_n - x_n)},$$

for $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, $\beta_2 + \beta_3 \neq 0$ and

$$u_n = z_n - \frac{f(z_n)}{f'(x_n)} \Big(\frac{f(x_n) - f(w_n)}{f(x_n) - 2f(w_n)} + \frac{1}{2} \frac{f(z_n)}{f(w_n) - 2f(z_n)}\Big)^2.$$

- Khan, Fardi and Sayevand's method [26] (KFS):

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= w_n - \frac{f^2(x_n)}{f^2(x_n) - 2f(x_n)f(w_n) + \beta_1 f^2(w_n)} \frac{f(w_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{1}{1 + \beta_2 q_n^2} \frac{f(z_n)}{K - C(w_n - z_n) - D(w_n - z_n)^2}, \end{aligned}$$

where $\beta_1, \beta_2 \in \mathbb{R}$, while $q_n = f(z_n)/f(x_n)$,

$$D = \frac{f'(x_n) - H}{(x_n - w_n)(x_n - z_n)} - \frac{H - K}{(x_n - z_n)^2},$$

$$C = \frac{H - K}{x_n - z_n} - D(x_n + w_n - 2z_n),$$

$$H = \frac{f(x_n) - f(w_n)}{x_n - w_n}$$

and

$$K = \frac{f(w_n) - f(z_n)}{w_n - z_n}.$$

- Chun and Lee's method [17] (CL):

$$\begin{aligned} w_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(w_n)}{f'(x_n)} \frac{1}{\big[1 - \frac{f(w_n)}{f(x_n)}\big]^2}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)} \cdot \frac{1}{H_n^2}, \end{aligned}$$

where

$$H_n = 1 - \frac{f(w_n)}{f(x_n)} - \frac{f(z_n)}{2f(x_n)} - \frac{f(z_n)}{2f(w_n)} - \frac{1}{2}\Big(\frac{f^2(w_n)}{f^2(x_n)} - \frac{f^3(w_n)}{f^3(x_n)}\Big).$$

Numerical results displayed in the following tables have been calculated for the values of real parameters suggested by authors of cited papers, i.e. $\beta = 1$ for BWR; $\beta_1 = 0, \beta_2 = 0$ for TP; $\beta_1 = 5, \beta_2 = -7$ for LW; $\beta_1 = 0, \beta_2 = 1, \beta_3 = 0$ for CTV; $\beta_1 = 1, \beta_2 = 1$ for KFS.

Four test examples have been employed to illustrate the numerical behavior of given methods.

Example 1. (Population growth problem) The test function has a form

$$f_1(x) = 1365 - 1000e^x - \frac{300}{x}(e^x - 1), \tag{14}$$

where the root is $\alpha \approx 0.05504622$. In fact, nonlinear function (14) is a particular case derived from the law of the population growth which is defined as a differential equation

$$\frac{dN(t)}{dt} = x \cdot N(t) + \eta,$$

where x is the birth rate, $N(t)$ is population at time, η is the immigration rate (see [27] for more details).

Example 2. (Van der Waals equation of state) This equation describes the behavior of the real gas with the respect to the two parameters α_1 and α_2, specific for each gas. The problem is to determine the volume V of the gas from the equation

$$PV^3 - (na_2P + nRT)V^2 + \alpha_1 n^2 V - \alpha_1\alpha_2 n^2 = 0$$

in terms of the remaining parameters. In [27] the researchers set the values for n, P, R and T, and considered the following function

$$f_2(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675,$$

with the desired simple root $\alpha = 1.72$, and the multiple root $\alpha = 1.75$ which is out of interest for this research.

Example 3. The problem of minimum insurance premium determination given by function

$$f_3(x) = 2e^{-\sqrt{x}}(\sqrt{x}+1) - 2e^{-\sqrt{x+1}}(\sqrt{1+x}+1) - e^{-1},$$

where the simple root is $\alpha \approx 0.541920$.

Example 4. The following standard nonlinear test function is taken from [17],

$$f_4(x) = x^4 + \sin\frac{\pi}{x^2} - 5,$$

with the simple root $\alpha = \sqrt{2}$.

Tables I-IV contain the numerical results for all presented eighth-order methods. The results are organized in three columns. The column named by "it" shows the number of iterations required to satisfy the stopping criterion

$$|x_{n+1} - x_n| + |f(x_n)| < 10^{-200}.$$

The next column $|f(x_3)|$ displays the absolute value of the function evaluated after fourth iteration, i.e. after 12 function/derivative evaluations. The last column presents computational order of convergence (COC) given by

$$\text{COC} = \frac{\log|f(x_n)/f(x_{n-1})|}{\log|f(x_{n-1})/f(x_{n-2})|}.$$

Table I. Numerical results for function $f_1(x)$, $x_0 = 0.1$

method	it	$\lvert f(x_3)\rvert$	COC
BWR	4	$7.247 \cdot 10^{-821}$	8.0000
TP	4	$2.300 \cdot 10^{-781}$	8.0000
LW	4	$5.089 \cdot 10^{-853}$	8.0000
CTV	4	$1.715 \cdot 10^{-894}$	8.0000
KFS	4	$1.677 \cdot 10^{-827}$	8.0000
CL	4	$1.015 \cdot 10^{-808}$	8.0000
SA_1	4	$6.762 \cdot 10^{-915}$	8.0000
SA_2	4	$8.462 \cdot 10^{-800}$	8.0000
SA_3	4	$2.401 \cdot 10^{-762}$	8.0000
$New_1, b = -0.5$	4	$1.227 \cdot 10^{-959}$	8.0000
$New_2, b = -0.5$	4	$2.729 \cdot 10^{-927}$	8.0000
$New_3, b = -0.5$	4	$1.792 \cdot 10^{-990}$	8.0000
$New_1, b = -1.5$	4	$2.297 \cdot 10^{-907}$	8.0000
$New_2, b = -1.5$	4	$8.801 \cdot 10^{-779}$	8.0000
$New_3, b = -1.5$	4	$3.006 \cdot 10^{-740}$	8.0000

Table II. Numerical results for function $f_2(x)$, $x_0 = 1.7$

method	it	$\lvert f(x_3)\rvert$	COC
BWR	5	$5.030 \cdot 10^{-117}$	8.0000
TP	5	$1.691 \cdot 10^{-70}$	8.0000
LW	5	$2.231 \cdot 10^{-65}$	8.0000
CTV	5	$1.630 \cdot 10^{-101}$	8.0000
KFS	5	$1.549 \cdot 10^{-93}$	8.0000
CL	5	$1.695 \cdot 10^{-100}$	8.0000
SA_1	5	$1.776 \cdot 10^{-190}$	8.0000
SA_2	5	$1.536 \cdot 10^{-193}$	8.0000
SA_3	5	$9.357 \cdot 10^{-190}$	8.0000
$New_1, b = 0.5$	5	$1.889 \cdot 10^{-191}$	8.0000
$New_2, b = 0.5$	5	$1.466 \cdot 10^{-197}$	8.0000
$New_3, b = 0.5$	5	$1.298 \cdot 10^{-195}$	8.0000
$New_1, b = 1.5$	5	$1.692 \cdot 10^{-193}$	8.0000
$New_2, b = 1.5$	4	$7.313 \cdot 10^{-208}$	8.0002
$New_3, b = 1.5$	4	$4.406 \cdot 10^{-213}$	8.0002

Table III. Numerical results for function $f_3(x)$, $x_0 = 0.25$

method	it	$\lvert f(x_3)\rvert$	COC
BWR	4	$1.186 \cdot 10^{-325}$	8.0000
TP	4	$2.179 \cdot 10^{-329}$	8.0000
LW	4	$4.506 \cdot 10^{-374}$	8.0000
CTV	4	$1.545 \cdot 10^{-429}$	8.0000
KFS	4	$6.283 \cdot 10^{-361}$	8.0000
CL	4	$7.165 \cdot 10^{-380}$	8.0000
SA_1	4	$1.243 \cdot 10^{-418}$	8.0000
SA_2	4	$6.089 \cdot 10^{-392}$	8.0000
SA_3	4	$4.391 \cdot 10^{-338}$	8.0000
$New_1, b = 0.5$	4	$2.066 \cdot 10^{-429}$	8.0000
$New_2, b = 0.5$	4	$2.546 \cdot 10^{-424}$	8.0000
$New_3, b = 0.5$	4	$4.776 \cdot 10^{-437}$	8.0000
$New_1, b = 1.5$	4	$9.669 \cdot 10^{-474}$	8.0000
$New_2, b = 1.5$	4	$5.576 \cdot 10^{-356}$	8.0000
$New_3, b = 1.5$	4	$1.185 \cdot 10^{-318}$	8.0000

Table IV. Numerical results for function $f_4(x)$, $x_0 = 1.2$

method	it	$\lvert f(x_3)\rvert$	COC
BWR	4	$1.969 \cdot 10^{-346}$	8.0000
TP	4	$2.111 \cdot 10^{-353}$	8.0000
LW	4	$6.321 \cdot 10^{-302}$	8.0000
CTV	4	$5.390 \cdot 10^{-334}$	8.0000
KFS	4	$1.645 \cdot 10^{-396}$	8.0000
CL	4	$9.304 \cdot 10^{-372}$	8.0000
SA_1	4	$7.437 \cdot 10^{-355}$	8.0000
SA_2	4	$2.268 \cdot 10^{-360}$	8.0000
SA_3	4	$6.629 \cdot 10^{-492}$	8.0000
$New_1, b = -0.5$	4	$7.518 \cdot 10^{-350}$	8.0000
$New_2, b = -0.5$	4	$5.797 \cdot 10^{-359}$	8.0000
$New_3, b = -0.5$	4	$1.482 \cdot 10^{-493}$	8.0000
$New_1, b = -1.5$	4	$9.437 \cdot 10^{-342}$	8.0000
$New_2, b = -1.5$	4	$4.149 \cdot 10^{-358}$	8.0000
$New_3, b = -1.5$	4	$2.996 \cdot 10^{-506}$	8.0000

All numerical experiments have been carried out by Mathematica 10 using *SetPrecision* function with 10000 significant digits.

IV. Conclusion

The new proposed iterative family (5) is of optimal eighth-order and satisfies the Kung-Traub conjecture. According to the COC values displayed in Tables I-IV, the numerical results clearly confirm theoretically derived eighth order as well. It is easy to observe that SA family (3) is a special case of family (5) for $b = 0$. By varying the values of parameter b, the convergence of SA methods can be improved in the sense that the sequence of approximations gets closer to the root within the same number of iterations (see for example Table II). On the other hand, in some cases for certain b values the convergence can be slower.

Although the new family shows very competitive results compared to other existing eighth-order methods, it cannot be said that some method is superior than others in general. However, choosing the best options for the value of parameter b could be worth for further research.

Acknowledgment

The authors acknowledge the financial support of the Ministry of Education, Science and Technological Development of the Republic of Serbia, within the Projects No. ON 174009 and TR 34014.

References

[1] A.M. Ostrowski, *Solution of Equations an Systems of Equations*, Academic Press, New York, 1960.

[2] S. Weerakoon and T. G. I. Fernando, *A variant of Newton's method with accelarated third-order convergence*, Applied Mathematics Letters 13, 87—93, 2000.

[3] A. Y. Özban, *Some new variants of Newton's method*, Applied Mathematics Letters 17, 677—682, 2004.

[4] T. Lukić, N. M. Ralević, *Geometric Mean Newton's Method for Simple and Multiple Roots*, Applied Mathematics Letters 21, 30—36, 2008.

[5] M. Frontini, E. Sormani, *Some variant of Newton's method with third-order convergence*, Applied Mathematics and Computation 140, 419—426, 2003 .

[6] J. Kou, Y. Li, X. Wang, *A modification of Newton method with third-order convergence*, Applied Mathematics and Computation 181, 1106—1111, 2006 .

[7] S.K. Parhi, D.K. Gupta, *A fourth order iterative method for solving nonlinear equations*, Applied Mathematics and Computation 203, 50—55, 2008.

[8] J.R. Sharma, R.K. Guha, *A family of modified Ostrowski methods with accelerated sixth order convergence*, Applied Mathematics and Computation 190, 111—115, 2007.

[9] H.T. Kung, J.F. Traub, *Optimal order of one-point and multipoint iteration*, J. ACM 21, 643—651, 1974

[10] P. Jarratt, *Some fourth order multipoint methods for solving equations*,

[11] P. Jarratt, *Some efficient fourth-order multipoint methods for solving equations*, BIT 9, 119—124, 1969.

[12] A.K. Maheshwari, *A fourth order iterative method for solving nonlinear equations*, Applied Mathematics and Computation 211, 383—391, 2009.

[13] J. Kou, Y. Li, X. Wang, *A coposite fourth-order iterative method for solving nonlinear equations*, Applied Mathematics and Computation 184, 471—475, 2007.

[14] C. Chun, *Some fourth-order iterative methods for solving nonlinear equations*, Applied Mathematics and Computation 195, 454—459, 2008.

[15] R.F. King, *A family of fourth order methods for solving nonlinear equations*, SIAM J. Numer. Anal. 10, 876—879, 1973.

[16] A. Cordero, J.R. Torregrosa, M.P Vassileva, *Three-step iterative methods with optimal eighth-order convergence*, Journal of Computational and Applied Mathematics 235, 3194—3194, 2011.

[17] C. Chun, M.Y. Lee, *A new optimal eighth-order family of iterative methods for the solution of nonlinear equations*, Applied Mathematics and Computation 223, 506—519, 2013.

[18] Y.H. Geum, Y.I. Kim, *An Optimal Family of Fast 16th-order Derivative-Free Multipoint Simple-Root Finders for Nonliear Equations*, J. Optim. Theory Appl., 2013.

[19] J.R. Sharma, H. Arora, *An efficient family of weighted-Newton methods with optimal eighth order convergence*, Applied Mathematics Letters 29, 1—6, 2008., 2014.

[20] X.Y. Wu, *A new continuation Newton-like method and its deformation*, Applied Mathematics and Computation 112, 75—78, 2000.

[21] M. Grau, J.L. Diaz-Barrero, *A technique to composite a modified Newtons method for solving nonlinear equations*, Ann. Univ. Buchar. Math. Ser. 2(LX), 53—61, 2011.

[22] W. Bi, H. Ren, Q. Wu, *Three-step iterative methods with eighth-order convergence for solving nonlinear equations*, J. Comput. Appl. Math. 225, 105—112, 2009.

[23] R.Thukral, M. S.Petković, *A family of three-point methods of optimal order for solving nonlinear equations*, J. Comput. Appl. Math. 233, 2278—2284, 2010.

[24] L. Liu, X.Wang, *Eighth-order methods with high efficiency index for solving nonlinear equations*, Applied Mathematics and Computation 215, 3449—3454, 2010.

[25] A.Cordero, J.R.Torregrosa, M.P.Vassileva, *Three-step iterative methods with optimal eighth-order convergence*, J. Comput. Appl. Math. 235, 3189—3194, 2011.

[26] Y.Khan, M. Fardi, K. Sayevand, *A new general eighth-order family of iterative methods for solving nonlinear equations*, Appl. Math. Lett. 25, 2262—2266, 2012.

[27] R. Behl, A.S. Alshomrani, S.S. Motsa *An optimal scheme for multiple roots of nonlinear equations with eighth-order convergence*, Journal of Mathematical Chemistry, 1—16, 2018.