Non-PDC Controller design for Uncertain Discrete-Time T-S Descriptor Models Subject to Input Saturation.

Ines Righi, Sabrina Aouaouda and Mohammed Chadli

February 20, 2020

Non-PDC Controller design for Uncertain Discrete-Time T-S Descriptor Models Subject to Input Saturation

Ines Righi, Sabrina Aouaouda, and Mohammed Chadli

Abstract— **This paper is concerned with the stabilization of uncertain discrete-time descriptor models subject to input saturation and external disturbances. The design control strategy is based on Takagi-Sugeno (T-S) approach and a non parallel distributed compensation (non-PDC) control law. To synthesis the fuzzy controller, the stability conditions are derived using non-quadratic Lyapunov functions with respect to the given saturation constraint on the control input. The optimization problem is formulated in terms of linear matrix inequalities (LMIs). Numerical examples illustrate the efficiency of the proposed approaches.**

I. INTRODUCTION

Recently, Takagi-Sugeno (T-S) models have been widely investigated to study nonlinear models [1]. Among nonlinear control theory, the T-S model-based approach has attracted great interest since it constitutes universal approximation of any smooth nonlinear function by a "blending" of some local linear system models. This method greatly facilitates observer/controller synthesis for complex nonlinear systems [2]. Based on this modeling technique, stability conditions have been obtained directly from Lyapunov methodology [3, 4]. For control design, the so-called parallel distributed compensation (PDC) has been the most commonly used scheme and remain to associate inferred state of output feedback to each local subsystem. Stabilization of T-S systems and the control design are investigated via the direct Lyapunov method. However, when fuzzy Lyapunov functions and parallel distributed compensation function (PDC) control are considered, the stabilization conditions are generally in terms of bilinear matrix inequalities especially for discrete-time T-S fuzzy [13]. To overcome such problem, a non-PDC control law can be applied based on non-quadratic Lyapunov function. Hence, more relaxed stabilization condition can be derived [3, 4-5]. The derived conditions are formulated into a set of linear matrix inequalities (LMI). These LMIs can be solved, when a solution exists, by classical convex optimization algorithms. In practice, many systems are physically described by nonlinear descriptor models. The T-S descriptor model representation has also the advantage to decrease the number of LMI constraints since it conserves nonlinearities in the left-hand side will keeping the original structure of the nonlinear model [7, 11-15].

I. Righi. and S. Aouaouda are with Faculty of Sciences and Technologies Univ. Souk Ahras-LEER- BP 1553.Souk-Ahras, 41000. Algeria. {i.righi, sabrina.aouaouda@univ-soukahras.dz}. M. Chadli is with University of Picardie Jules Verne, MIS (E.A.4290), 33, rue Saint-Leu, 80039 Amiens, France. e-mail: mchadli@u-picardie.fr

Usually real physical applications suffer from actuator saturation and/or sensor saturation. Thus, a great attention has been given to the control design of T-S models with input saturation constraints [8-9]. Among the most popular works dealing with saturated input constraints, the convexity based approach to the saturation function (see [9-10] and the references therein). The interest of this approach is to consider a bounded ellipsoidal symmetric region of stability solved by a set of LMIs. Moreover, descriptor design approaches has been recently studied in [11] to deal with the problem of input saturated T-S systems using a polytopic representation of the saturation function. These design control approaches are based on state feedback or dynamic output feedback with anti-windup (AW) mechanisms.

In the present paper, stability analysis for uncertain discrete-time T-S descriptor models subject to input saturation and unknown disturbances is proposed. New LMI conditions are derived based on two Lyapunov functions with the Finsler's lemma which allows decoupling the control law from the Lyapunov function. The $\mathcal{L}_2$-gain performance is used to attenuate the extragenous disturbances and the derived conditions of asymptotic stability in the presence of input saturation are established and solved by means of LMI convex optimization.

This paper is organized as follows: Section II provides some useful notation and properties, it also introduces the uncertain discrete-time T-S descriptor model; Section III presents the LMI-based controller design for discrete-time T-S descriptor models subject to: uncertainties, input saturation and disturbances; Finally in section IV designed examples are given to demonstrate the effectiveness of the proposed approaches.

II. NOTATION AND PROBLEM STATEMENT

Given a set of nonlinear functions $h_i(.) \geq 0, i \in \{1 \dots . r\}$ having the convex sum property, $\sum_{i=1}^{r} h_i(.) = 1$; a shorthand notation will be used in the sequel to represent convex sum of matrix expressions: $Y_h = \sum_{i=1}^{r} h_i\,(z(k))Y_i$ and $Y_v = \sum_{i=1}^{r} v_k\,(z(k))Y_k$ for single convex sum; $Y_{h+} = \sum_{i=1}^{r} h_l\,(z(k+1))Y_l$ for a delayed convex sum; $Y_{h-} = (\sum_{i=1}^{r} h_i z(k)Y_i)^{-1}$ for the inverse of a convex sum; and, $Y_{hh} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\big(z(k)\big)h_j\,(z(k))Y_{ij}$ for a doubled rested convex sum. $\mathcal{H}(A)$ denotes the Hermitian of the matrixA, i.e. $\mathcal{H}(\mathrm{A}) = \mathrm{A} + \mathrm{A}^{\mathrm{T}}$.For a vector x, x_k denotes its k_th entry, and x_{k+} denotes $x(k+1)$. $\mathbb{N}_{\mathrm{r}}$ Denotes the set $\{1, 2, \cdots, r\}$, I denotes the identity matrix.(*) stands for the terms deduced by symmetry in symmetric block matrices.

The uncertain discrete-time T-S fuzzy model in the descriptor form subject to input saturation and external disturbances is given by following state equations:

$$\begin{cases} E_v x_{k+} = (A_h + \Delta A_h)x_k + (B_h + \Delta B_h)sat\big(u(k)\big) + B_\omega \omega_k \\ y_k = Cx \end{cases} \quad (1)$$

where $x \in \Re^n$ is the state vector, $u \in \Re^m$ is the control input vector, k is the current sample. Matrices A_i and B_i, $i \in \mathbb{N}_r$ represent the *i-th* linear right-hande side model and E_v, $k \in \{1 \dots \dots r_e\}$, represents the *k-th* linear left-hand side model of T-S descriptor models. In the proposed work we suppose that E_v is regular matrix because it contains the inertia matrix which is motivated by mechanical systems [12]. The membership functions $h_i(z(k))$ satisfy the convex sum property and depend on the known premise variables. $\Delta A_i(t) \in \Re^{nxn}$ and $\Delta B_i(t) \in \Re^{nxm}$ contains the bounded uncertain terms which can be rewritten as:

$\Delta A_i(t) = H_{ai}\mathfrak{D}_{ai}(t)N_{ai}$ and $\Delta B_i(t) = H_{bi}\mathfrak{D}_{bi}(t)N_{bi}$ with $H_{ai}, H_{bi}, N_{ai}, and\ N_{bi}$ are known constant matrices and, $\mathfrak{D}_{ai}(t)\ and\ \mathfrak{D}_{bi}(t)$ are unknown matrices functions bounded as: $\mathfrak{D}^T_{\varepsilon i}(t)\mathfrak{D}_{\varepsilon\theta}(t) \le I$, for all index $\varepsilon = a, b.$

The dead-zone nonlinearity $\psi(.): \Re^{n_u} \to \Re^{n_u}$ is defined by:

$$\boldsymbol{\psi(u_k) = u_k - sat(u_k)} \quad (2)$$

However the saturation function $\boldsymbol{sat}: \boldsymbol{\Re}^{n_u} \to \boldsymbol{\Re}^{n_u}$ is written as:

$$\begin{cases} sat(u_k) = [sat_1(u_{1k}) \dots \dots sat_t(u_{tk}) \dots \dots sat_{n_u}(u_{n_{uk}}) \\ sat_t(u_{tk}) = sign(u_{tk})\, min(|u_{tk}|, u_t^{max}) \end{cases} \quad (3)$$

with $u_t^{max} > 0$, designate the saturation level.

Remark 1: In control applications, actuator saturation or control input saturation both in magnitude and rate usually degrades the performance of the closed-loop system, and leads to large overshoot, if the controller is designed without considering these kinds of nonlinearity. Recently, in the closed loop control system, the nonlinear system, behavior of the input saturation has been investigated as a convex combination of 2^m linear models in [10, 11].

In this work, in order to derive relaxed LMI conditions, the following lemmas will be introduced to reduce significantly the computational complexity.

Relaxation Lemma [14]: Let T_{ij}^k be matrices of appropriate dimensions, then for $i, j \in \{1 \dots . r\}, k \in \{1 \dots . . r_e\}$:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} h_i(z(k))\, h_j(z(k)) v_k(z(k))$$

$$T_{ij}^k < 0$$

$$\frac{2}{r-1}T_{ii}^k + T_{ij}^k + T_{ji}^k < 0, i \ne j \quad (4)$$

Finsler's Lemma [7]. Let $x \in \Re^n, \varrho = \varrho^T \in \Re^{n\times n}$ and $R \in \Re^{m\times n}$ such that $rank(R) < n$, the following expressions are equivalent:

a) $x^T\varrho x < 0 \quad \forall x: \{x \in \Re^n\ ; x \ne 0, Rx = 0\}$.

b) $\exists\ \Theta \in \Re^{mxn}$: $\varrho + \Theta R + R^T\Theta^T < 0$.

Property 1. Let $X = X^T > 0$, and Y matrices of the appropriate size the following expressions holds:

$$(Y - X)^T X^{-1}(Y - X) \ge 0 \Leftrightarrow Y^T X^{-1} Y \ \ge Y + Y^T - X$$

Lemma 1 [16]: Given matrices $F_{jk} \in \Re^{m\times n}$, $H_{jk} \in \Re^{n\times n}$ and $W_{jk} \in \Re^{m\times n}$, for $i, j, k \in \mathbb{N}_r, l \in \mathbb{N}_m$, let us define the following set:

$$\mathcal{P}_u = \{x \in \Re^n: |(G_{jk}\chi_{jk}^{-1} - W_{jk}\chi_{jk}^{-1})x_k| \le u_{max(l)}\} \quad (5)$$

If $x \in \mathcal{P}_u$, then $\psi(u_k)^T S_{jk}^{-1}[\,\psi(u_k) - (W_{jk}\chi_{jk}^{-1})x_k] \le 0$

holds of any positive diagonal matrices $S_{jk} \in \Re^{m\times m}$, and for any scalar function $\chi_{ij}, i, j \in \mathbb{N}_r$, satisfying the convex sum property.

Assumption 1. The validity domain Ω_x of the system (1) is defined by:

$$\Omega_x = \{x \in \Re^n: \mathcal{Q}_m^T x \le 1,\ \ m \in \mathbb{N}_q\} \quad (6)$$

where the vectors $\mathcal{Q}_m \in \Re^n$ are corresponding to the state constraints of system (1).

III. Main Results

In this section, the objective is to design a non-PDC control law guarantying the desired control performance and the stability of the closed loop system, despite the presence of input control saturation and external disturbances. Accordingly, the proposed non-PDC controller law is given as follows:

$$u_k = G_{hv}\chi_{(.)}^{-1}x_k \quad (7)$$

In the following we consider that: $\mathrm{A_h} + \Delta\mathrm{A_h} = \mathbb{A}$, $\mathrm{B_h} + \Delta\mathrm{B_h} = \mathbb{B}$, $\mathrm{E_v} = \mathrm{E}$. The combination of the uncertain T-S descriptor model (1) with the control law (7) and the definition (2) yields:

$$\begin{cases} Ex_{k+} = \mathbb{A}x_k + \mathbb{B}\left(G_{hv}\chi_{(.)}^{-1}x_k - \psi(u_k)\right) + B_\omega\omega_k \\ y_k = Cx_k \end{cases} \quad (8)$$

Expression (8) can be rewritten as an equality constraint as:

$$\begin{bmatrix}\mathbb{A} + \mathbb{B}G_{hv}\chi_{(.)}^{-1} & -E & -\mathbb{B} & B_\omega\end{bmatrix}\begin{bmatrix}x_k \\ x_{k+} \\ \psi(u_k) \\ \omega_k\end{bmatrix} = 0 \quad (9)$$

To derive the stability conditions, two different Lyapunov functions will be considered [7]:

- $V(x_k) = x_k^T P_h^{-1} x_k\ with\ P_h = P_h^T > 0, P_h^{-1} = X_h$
- $V(x_k) = x_k^T \chi_h^{-T} P_h\, \chi_h^{-1} x_k, with\ P_h = P_h^T > 0$

A. Case 1

The variation of the Lyapunov function in case 1 is calculated as:

$$V(x_k) = x_{k+}^T X_{h+} x_{k+} - x_k^T X_h x_k < 0 \quad (10)$$

By taking into consideration Lemma 1, the inequality (10) becames:

$$x_{k+}^T X_{h+} x_{k+} - x_k^T X_h x_k - 2.\psi(u_k)^T S_{hv}^{-1}\psi(u_k) + \psi(u_k)^T S_{hv}^{-1} W_{hv}\chi_{hv}^{-1}x_k + y_k^T y_k - \gamma\omega_k^T\omega_k < 0 \quad (11)$$

where the $\mathcal{L}_2$ −norm of the output signal y_k is bounded as follows:

$$\|y_k\|_2 \leq \gamma \|\omega_k\|_2, \quad \forall k \geq 0. \tag{12}$$

Consequently, the inequality (11) can be written as:

$$\begin{bmatrix} x_k \\ x_{k+} \\ \psi(u_k) \\ \omega_k \end{bmatrix}^T \begin{bmatrix} -(X_h - C^T C) & 0 & 0 & 0 \\ 0 & X_{h+} & 0 & 0 \\ S_{hv}^{-1} W_{hv} G_{hv}^{-1} & 0 & -2S_{hv}^{-1} & 0 \\ 0 & 0 & 0 & -\gamma I \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+} \\ \psi(u_k) \\ \omega_k \end{bmatrix} < 0 \tag{13}$$

Via the Finsler's Lemma, equality (9) and inequality (13) results in:

$$\begin{bmatrix} -(X_h - C^T C) & 0 & 0 & 0 \\ 0 & X_{h+} & 0 & 0 \\ W_{hv} & -\mathbb{B}^T N^T & -2.S_{hv}^T & 0 \\ 0 & B_\omega^T & 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} M \\ N \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbb{A} + \mathbb{B} G_{hv} \chi_{(.)}^{-1} & -E & -\mathbb{B} & B_\omega \end{bmatrix} + (*) < 0 \tag{14}$$

where matrices $M \in \Re^{nxn}$ and $N \in \Re^{nxn}$are free matrices fixed later on. Now let $\chi_{(.)} = \chi_{hv}$, two results can be obtained depending on different congruence transformation of (14). The first one is stated in the following Lemma:

Lemma 2. The closed-loop uncertain T-S descriptor model (8) is asymptotically stable if there exist matrices $P_j = P_j^T > 0, \chi_{jk}, G_{jk}, W_{jk}$, a matrix $Q_m \in \mathbb{R}^n, m \in \mathbb{N}_q$ and positive diagonal matrices $S_{jk} \in \mathbb{R}^{m\times m}$, for $i, j, l \in \{1 \dots \dots r\}, k \in \{1 \dots \dots r_e\}$, the scalars $\gamma, \partial_1, \partial_2$ such that conditions 4 are satisfied with

$$T_{ijl}^k = \begin{bmatrix} \Psi_{ijl}^{k11} & (*) \\ \Psi_{ij}^{21} & -\Psi^{22} \end{bmatrix} \tag{15a}$$

where

$$\Psi_{ijl}^{k11} = \begin{bmatrix} -\mathcal{H}(\chi_{jk}) + P_j & (*) & (*) & (*) & (*) \\ C\chi_{jk} & I & (*) & (*) & (*) \\ \begin{pmatrix} A_i \chi_{jk} + B_i G_{jk} \\ +\partial_1 H_{ai} H_{ai}^T \\ +\partial_2 H_{bi} H_{bi}^T \end{pmatrix} & 0 & -\mathcal{H}(P_l E^T) + P_l & (*) & (*) \\ W_{jk} & 0 & \begin{pmatrix} -S_{jk}^T B_i \\ -\partial_3 H_{bi} H_{bi}^T \end{pmatrix} & -2S_{jk}^T & (*) \\ 0 & 0 & B_\omega^T & 0 & -\gamma I \end{bmatrix} \tag{15b}$$

$$\Psi_{ij}^{21} = \begin{bmatrix} N_{ai}\chi_{jk} & 0 & 0 & 0 & 0 \\ N_{bi} G_{jk} & 0 & 0 & 0 & 0 \\ 0 & 0 & N_{bi} S_{jk} & 0 & 0 \end{bmatrix} \tag{15c}$$

$$\Psi^{22} = diag(\partial_1 I \quad \partial_2 I \quad -\partial_2 I) \tag{15d}$$

and

$$\begin{bmatrix} -P_j & (*) \\ Q_m P_j & -1 \end{bmatrix} \leq 0, m \in \mathbb{N}_q, \ j \in \mathbb{N}_r \tag{16}$$

$$\begin{bmatrix} -\chi_{jk}{}^T - \chi_{jk} + P_j & (*) \\ G_{jk(t)} - W_{jk(t)} & -(u_t^{max})^2 \end{bmatrix} \leq 0, t \in \mathbb{N}_m, j, k \in N_r \quad 17)$$

Proof. From inequality (16) it can be deduced that $x \in \Omega_x$. Furthermore, if condition (16) is satisfied, then it follows clearly that matrices $\chi_{jk}, j, k \in \mathbb{N}_r$, are regular since $P_j > 0$. Besides, using Schur complement lemma [15] and matrix property 1, it can be deduced from (17) that

$$G_{jk}^T P_j G_{jk} - \frac{(G_{jk(t)} - W_{jk(t)})^T (G_{jk(t} - W_{jk(t)})}{(u_t^{max})^2} \geq 0 \tag{18}$$

Pre and post-multiplying (2) with χ_{jk}^{-T} yields:

$$P_j - \frac{(G_{jk(t)}\chi_{jk}^{-1} - W_{jk(t)}\chi_{jk}^{-1})^T (G_{jk(t)}\chi_{jk}^{-1} - W_{jk(t)}\chi_{jk}^{-1})}{(u_t^{max})^2} \geq 0 \tag{19}$$

Then, it is easily observed that condition (19) implies the inclusion $\Omega_x \subseteq \mathcal{P}_u$. Now, by using the congruence lemma property with the full rank matrix $diag\begin{pmatrix} \chi_{hv}^T & P_{h+} & S_{hv}^T & I \end{pmatrix}$, (14) yields:

$$\begin{bmatrix} -\chi_{hv}^T (X_h - C^T C)\chi_{hv} & 0 & 0 & 0 \\ 0 & P_{h+} X_{h+} P_{h+} & 0 & 0 \\ W_{hv} & -S_{hv}^T \mathbb{B}^T & -2S_{hv}^T & 0 \\ 0 & B_\omega^T & 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} \chi_{hv}^+ M \\ P_{h+} N \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbb{A}\chi_{hv} + \mathbb{B} G_{hv} & -E\chi_{hv} & -\mathbb{B} S_{hv} & B_\omega \end{bmatrix} + (*) < 0 \tag{20}$$

The main objective is to design a relaxed LMI optimization problem. For that a best choice is to consider $M = 0$ and $N = X_{h+}$ then (20) yields if:

$$\psi_{hv} + \Delta\psi_{hv} < 0 \tag{21a}$$

with

$$\psi_{hv} = \begin{bmatrix} -\chi_{hv}^T (X_h - C^T C)\chi_{hv} & (*) & (*) & (*) \\ A_h \chi_{hv} + B_h G_{hv} & -\mathcal{H}(P_{h+} E^T) + P_{h+} & (*) & (*) \\ W_{hv} & -S_{hv}^T B_h^T & -2S_{hv}^T & (*) \\ 0 & B_\omega^T & 0 & -\gamma I \end{bmatrix}$$

and $$\Delta\psi_{hv} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \Delta A_h \chi_{hv} + \Delta B G_{hv} & 0 & 0 & 0 \\ 0 & -S_{hv}^T \Delta B_h^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Recall that for any matrices $\mathcal{G}, \mathcal{K}$ and $\mathfrak{D}(t)$ of appropriate dimension satisfying $\mathfrak{D}^T(t)\mathfrak{D}(t) \leq I$ and any positive scalar ∂**,** the following holds

$$\mathcal{G}\mathfrak{D}(t)\mathcal{K}^T + \mathcal{K}\mathfrak{D}^T(t)\mathcal{G}^T \leq \partial\mathcal{G}\mathcal{G}^T + \partial^{-1}\mathcal{K}\mathcal{K}^T \tag{21b}$$

Now, using uncertainties definitions, and by means of the previous inequality (21b), it can be stated that:

$$\Delta\psi_{hv} < \begin{bmatrix} 0 & 0 & 0 & 0 \\ \Delta^{21} & 0 & 0 & 0 \\ 0 & \begin{pmatrix} -\partial_3^{-1} S_{hv}^T N_{bh}^T N_{bh} S_{hv} \\ -\partial_3 H_{bh} H_{bh}^T \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{21c}$$

with

$$\Delta^{21} = \begin{pmatrix} \partial_1^{-1} \chi_{hv}^T N_{ah}^T N_{ah} \chi_{hv} + \partial_2^{-1} G_{hv}^T N_{bh}^T N_{bh} G_{hv} \\ +\partial_1 H_{ah} H_{ah}^T + \partial_2 H_{bh} H_{bh}^T \end{pmatrix}$$

Finally, applying property1, the relaxation lemma and Schur complement lemma to (21a) by considering (21c), conditions (15) holds. This ends the proof.

Remark 2. A more general result can be reached by multiplying by $diag\left(\chi_{hv}^T \quad G_{hhh+}^T \quad S_{hv}^T \quad I\right)$, on the left hand side and by its transpose, on the right hand side of (14), gives:

$$\begin{bmatrix} -\chi_{hv}^T(X_h - C^TC)\chi_{hv} & 0 & 0 & 0 \\ 0 & G_{hhh+}^T X_{h+} G_{hhh+} & 0 & 0 \\ W_{hv} & -S_{hv}^T B^T & -2S_{hv}^T & 0 \\ 0 & B_\omega^T & 0 & -\gamma I \end{bmatrix} +$$

$$\begin{bmatrix} \chi_{hv}^+ M \\ G_{hhh+} N \\ 0 \\ 0 \end{bmatrix} [A\chi_{hv} + BG_{hv} \quad -EG_{hhh+} \quad -B_h S_{hv} \quad B_\omega] + (*) < 0 \quad (22)$$

Note that a new matrix G_{hhh+}^T is introduced, thus adding extra degree of freedom to the inequality. Therefore, the following theorem can be stated:

Theorem 1. Given the uncertain T-S descriptor system (1) whose validity domain is characterized by a matrix $Q_m \in \Re^n, m \in \mathbb{N}_q$. If there exist positive definite matrices $P_j \in \Re^{n\times n}$, positive diagonal matrices $S_{jk} \in \Re^{m\times m}$, and matrices $\chi_{jk} \in \Re^{n\times n}$, $G_{jk} \in \Re^{m\times n}, W_{jk} \in \Re^{m\times n}, i,j,k \in \mathbb{N}_r$, and the scalars $\gamma, \partial_1, \partial_2$, such that conditions 4 are satisfied with:

$$T_{ijl}^k = \begin{bmatrix} \phi_{ijl}^{k11} & (*) \\ \Psi_{ij}^{21} & -\Psi^{22} \end{bmatrix} \quad (23)$$

with

$$\phi_{ijl}^{k11} = \begin{bmatrix} -\mathcal{H}(\chi_{jk}) + P_j & (*) & (*) & (*) & (*) & (*) \\ C\chi_{jk} & I & (*) & (*) & (*) & (*) \\ \begin{pmatrix} A_i\chi_{jk} + B_iG_{jk} \\ +\partial_1 H_{ai}H_{ai}^T \\ +\partial_2 H_{bi}H_{bi}^T \end{pmatrix} & 0 & -\mathcal{H}(G_{ijl}{}^T E^T) & (*) & (*) & (*) \\ 0 & 0 & G_{ijl} & -P_l & (*) & (*) \\ W_{jk} & 0 & \begin{pmatrix} -S_{jk}^T B_i \\ -\partial_3 H_{bi}H_{bi}^T \end{pmatrix} & 0 & -2S_{jk}^T & (*) \\ 0 & 0 & B_\omega^T & 0 & 0 & -\gamma I \end{bmatrix} \quad (24)$$

Ψ_{ij}^{21} and Ψ^{22} are given by (15c) and (15d) respectively.

and

$$\begin{bmatrix} P_j & (*) \\ Q_m P_j & 1 \end{bmatrix} \geq 0, m \in \mathbb{N}_q, j \in \mathbb{N}_r \quad (25)$$

$$\begin{bmatrix} -\chi_{jk}{}^T - \chi_{jk} + P_j & (*) \\ G_{jk(t)} - W_{jk(t)} & -(u_t^{max})^2 \end{bmatrix} \leq 0, t \in \mathbb{N}_m, j,k \in N_r \quad (26)$$

Proof. The result of Theorem 1 is derived from the proof of Lemma 2 by choosing $M = 0$ and $N = G_{hhh+}^{-T}$. Thus, this proof is omitted here for brevity.

.

B. Case 2

Consider $\chi_{(.)} = \chi_h$ in (7). Then the variation of the Lyapunov function in case 2 is calculated as:

$$\Delta V(x_k) = x_{k+}^T \chi_{h+}^{-T} P_{h+} \chi_{h+}^{-1} x_{k+} - x_k^T \chi_h^{-T} P_h \chi_h^{-1} x_k < 0 \quad (27)$$

By considering lemma1, the inequality (27) becames:

$$\Delta V(x_k) - 2\psi(u_k)^T S_{hv}^{-1}\psi + \psi(u_k)^T S_{hv}^{-1} W_{hv} \chi_h^{-1} x_k + y_k^T - \gamma\omega_k^T\omega_k < 0 \quad (28)$$

By developing inequality (28), with respect to the Finsler's Lemma, and expressions (9) one can obtain:

$$\begin{bmatrix} -\chi_h^{-T}(P_h - C^TC)\chi_h^{-1} & 0 & 0 & 0 \\ 0 & \chi_{h+}^{-T} P_{h+} \chi_{h+}^{-1} & 0 & 0 \\ S_{hv}^{-1} W_{hv} \chi_h^{-1} & -S_{hv}^T \mathbb{B}^T & -2S_{hv}^{-1} & 0 \\ 0 & B_\omega^T & 0 & -\gamma I \end{bmatrix} +$$

$$\begin{bmatrix} M \\ N \\ 0 \\ 0 \end{bmatrix} \left[\mathbb{A} + \mathbb{B} G_{hv} \chi_{(.)}^{-1} \quad -E \quad -\mathbb{B} \quad B_\omega\right] + (*) < 0 \quad (29)$$

Now, using the property of congruence with $diag\left(-\chi_h^T \quad \chi_{h+}^T \quad S_{hv}^T \quad I\right)$, (29) yields:

$$\begin{bmatrix} -(P_h - C^TC) & 0 & 0 & 0 \\ 0 & P_{h+} & 0 & 0 \\ S_{hv}^{-1} W_{hv} H_h^{-1} & -S_{hv}^T \mathbb{B}^T & -2S_{hv}^{-1} & 0 \\ 0 & B_\omega^T & 0 & -\gamma I \end{bmatrix} + \quad (30)$$

$$\begin{bmatrix} H_h^T M \\ H_{h+}^T N \\ 0 \\ 0 \end{bmatrix} [\mathbb{A}\chi_h + \mathbb{B}F_{hv} \quad -E\chi_{h+} \quad -\mathbb{B}S_{hv} \quad B_\omega] + (*) < 0$$

Accordingly, with respect to the same development of lemma 2 LMI conditions, the following theorem is obtained and provides conditions which allow the synthesis of the stabilization non-PDC controller satisfying the system performance described in the previous section.

Theorem 2. Given the uncertain T-S descriptor system (1) whose validity domain is characterized by a matrix $Q_m \in \Re^n, m \in \mathbb{N}_q$. If there exist positive definite matrices $P_j \in \Re^{n\times n}$, positive diagonal matrices $S_{jk} \in \Re^{m\times m}$, and matrices $\chi_j \in \Re^{n\times n}$, $G_{jk} \in \mathbb{R}^{m\times n}, W_{jk} \in \Re^{m\times n}, i,j,k \in \mathbb{N}_r$, and positive scalar $\gamma, \partial_1, \partial_2$, such that conditions 4 are satisfied with:

$$T_{ijl}^k = \begin{bmatrix} \phi_{ijl}^{k11} & (*) \\ \Psi_{ij}^{21} & -\Psi^{22} \end{bmatrix} \quad (31)$$

with

$$\Psi_{ijl}^{k11} = \begin{bmatrix} -P_j & (*) & (*) & (*) & (*) \\ C\chi_j & I & (*) & (*) & (*) \\ \begin{pmatrix} A_i\chi_j + B_i G_{jk} \\ +\partial_1 H_{ai} H_{ai}^T \\ +\partial_2 H_{bi} H_{bi}^T \end{pmatrix} & 0 & -\mathcal{H}(\chi_l^T E^T) + P_l & (*) & (*) \\ W_{jk} & 0 & \begin{pmatrix} -S_{jk}^T B_i \\ -\partial_3 H_{bi} H_{bi}^T \end{pmatrix} & -2S_{jk}^T & (*) \\ 0 & 0 & B_\omega^T & 0 & -\gamma I \end{bmatrix}$$

Ψ_{ij}^{21} and Ψ^{22} are given by (15c) and (15d) respectively.

and

$$\begin{bmatrix} -P_j & (*) \\ Q_m P_j & -1 \end{bmatrix} \leq 0, m \in \mathbb{N}_q,\ j \in \mathbb{N}_r \tag{32}$$

$$\begin{bmatrix} -\chi_{jk}{}^T - \chi_{jk} + P_j & (*) \\ G_{jk(t)} - W_{jk(t)} & -(u_t^{max})^2 \end{bmatrix} \leq 0, t \in \mathbb{N}_m, j,k \in N_r \tag{33}$$

Proof. The result of Theorem 2 is derived from the proof of Lemma 2 by choosing $M = 0$ and $N = \chi_{h+}^{-T} = P_{h+}$. Thus, this proof is omitted here for brevity.

IV. ILLUSTRATIVE EXAMPLES

In this section, the proposed solutions are illustrated via the following two numerical examples.

Example 1. Consider the uncertain T-S descriptor model (1), with: $r = r_e = 2$ and a = -1.7 ; $b = -0.4$; $c = 0.2172$.

$$A_1 = \begin{bmatrix} 0 & 0.5 \\ -1.5 & -3 + \left(\frac{b}{2}\right) * (1 - c) \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ \frac{a}{2} - 2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0.5 \\ -1.5 & -3 + b * (1 - c) \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ \frac{a}{2} - 2 \end{bmatrix}$$

$E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0.5 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & -1 \\ 1 & 0.5 \end{bmatrix}$, $B_w = \begin{bmatrix} 0.05 \\ 0 \end{bmatrix}$ and C = [0 1].

$$H_{a1} = H_{a2} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix};\ N_{a1} = \begin{bmatrix} 0 & 0.5 \\ 0 & \left(\frac{b}{2}\right) * (1 + c) \end{bmatrix}$$

$$N_{a2} = \begin{bmatrix} 0 & 0.5 \\ 0 & b * (1 + c) \end{bmatrix},\ \Delta_{a1} = \Delta_{a2} = 0.3 * \cos(2 * \mathrm{t}).$$

$H_{b1} = H_{b2} = [0 \quad 1]$; $\Delta_{b1} = \Delta_{b2} = 0.2 * \sin(3 * t)$

$\mathrm{N}_{b1} = \mathrm{N}_{b2} = \frac{1}{2} * a$. The membership functions are defined as follows:

$$v_{1k} = \frac{cos(x_{2k})^2 + 2}{4}, v_{2k} = 1 - v_{1k},$$

$$h_{1k} = \frac{sin(x_{1k})^2}{4}, h_{2k} = 1 - h_{1k}$$

The MFs satisfy the convex-sum property on the compact set $\Delta = \{x_k : |x_{1k}| \leq 2, |x_{2k}| \leq 2\}$, and $u^{max} = 0.05$. The uncertain T-S descriptor system (1) is subject to amplitude-bounded disturbance $\omega(t)$ defined by $\omega(t) = 0.1 * \sin(10 * t)$. For this model, only the conditions of Theorem1 are feasible, i.e., conditions in Theorem 2 are unfeasible. Solving the optimization LMI problem defined by Theorem 1 leads to:

$P_1 = \begin{bmatrix} 0.64 & -0.26 \\ -0.26 & 3.09 \end{bmatrix}, P_2 = \begin{bmatrix} 0.69 & 0.36 \\ 0.36 & 4.76 \end{bmatrix}$

$H_{11} = H_{12} = \begin{bmatrix} 2.15 & 6.89 \\ 3.24 & 3.87 \end{bmatrix}, H_{21} = H_{22} = \begin{bmatrix} 4.41 & -3.06 \\ -4.13 & 3.13 \end{bmatrix},$

$F_1 = F_2 = [-1.70 \quad -3.45]$, $F_3 = F_4 = [0.24 \quad -0.79]$.

$\gamma = 1.42$ and $\partial_1 = \partial_2 = 1$. Simulation results were obtained with initial conditions $x(0) = [0.5 \ - \ 0.25]^T$ are shown in Figure 1 and 2. The obtained results illustrate the effectiveness of the proposed approach for the studied example.

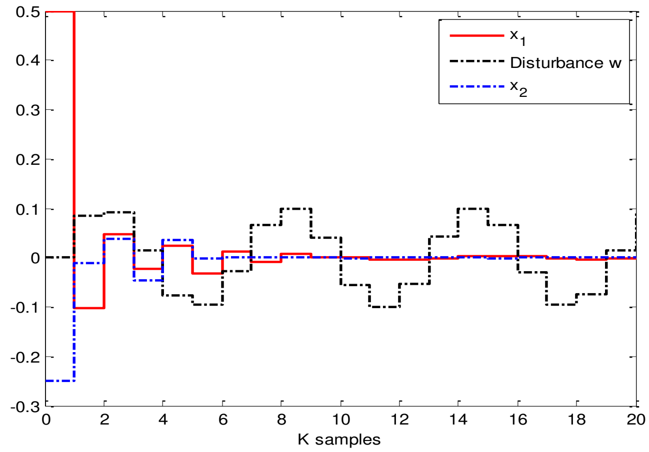

Figure 1. Closed loop responses in the presence of input saturation and disturbances.

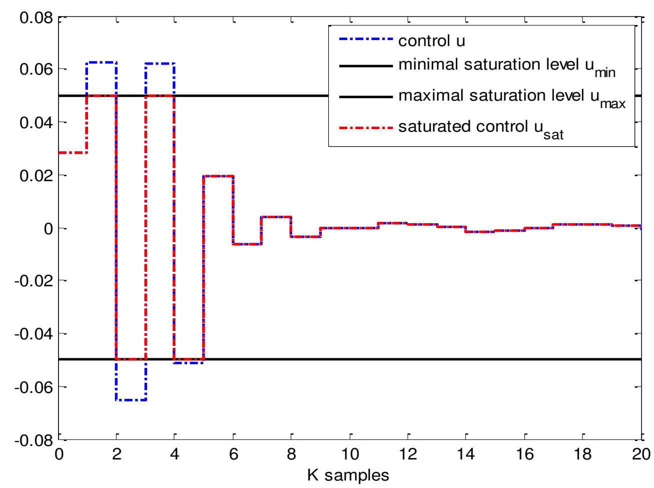

Figure 2. Control input.

Example 2. Consider an uncertain discrete-time T-S descriptor model as in (1) with $\mathrm{r} = \mathrm{r_e} = 2$, a = -1.7, $b = -0.4$, $c = 0.2172$ and

$\mathrm{A}_1 = \begin{bmatrix} 1 & 2 \\ -1.5 & -3 + \mathrm{b} * (1 + \mathrm{c}) \end{bmatrix}, \mathrm{A}_2 = \begin{bmatrix} 1 & 1 \\ -1.5 & -3 + \mathrm{b} \end{bmatrix}$

$\mathrm{B}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathrm{B}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathrm{E}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \mathrm{E}_2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

$\mathrm{B_w} = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix}$, $C = [0 \ \ 1]$. $H_{a1} = H_{a2} = \begin{bmatrix} 1 & 0.5 \\ 1 & 1 \end{bmatrix}$,

$N_{a1} = N_{a2} = \begin{bmatrix} 0 & 0.5 \\ 1 & 1.75 \end{bmatrix}$, $\Delta_{a1} = \Delta_{a2} = 0.8 * \cos(3 * t)$

$H_{b1} = H_{b2} = [0 \quad 1]$, $\Delta_{b1} = \Delta_{b2} = 0.25 * \sin(2 * t)$

$N_{b1} = N_{b2} = \frac{1}{2} * a$. The MFs are defined the same as in Example 1 and satisfy the convex-sum property on the compact set $\Delta = \{x_k : |x_{1k}| \le 1, |x_{2k}| \le 1\}$, and $u^{max} = 0.05$. For this system LMI conditions in Theorem 1 are unfeasible, while applying LMI Theorem 2 gives the following gain matrices:

$P_1 = \begin{bmatrix} 6.73 & -0.44 \\ -0.44 & 0.29 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0.89 & 1.60 \\ 1.60 & 0.62 \end{bmatrix}$

$H_1 = \begin{bmatrix} 0.98 & -0.53 \\ -0.47 & 0.77 \end{bmatrix}$, $H_2 = \begin{bmatrix} 0.91 & 0.79 \\ -0.32 & 0.77 \end{bmatrix}$,

$F_1 = F_2 = [-0.74 \quad -0.55]$,

$F_3 = F_4 = [-0.20 \quad -0.44]$. Simulation results with initial conditions $x(0) = [0.7 \quad -0.25]^T$ are depicted in Figure 3 and 4.

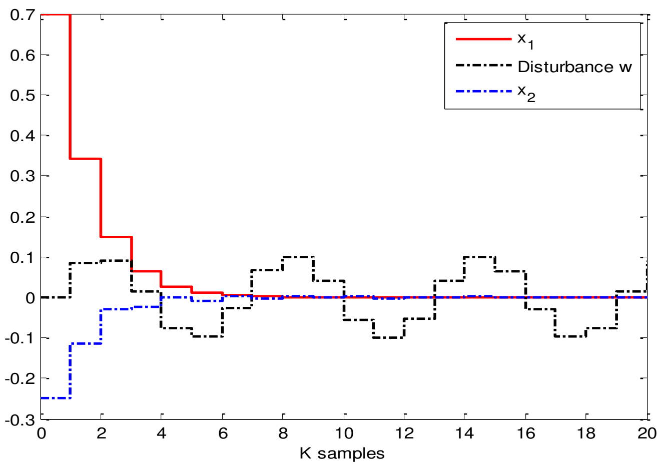

Figure 3. Uncertain system responses in the presence of input saturation and disturbances

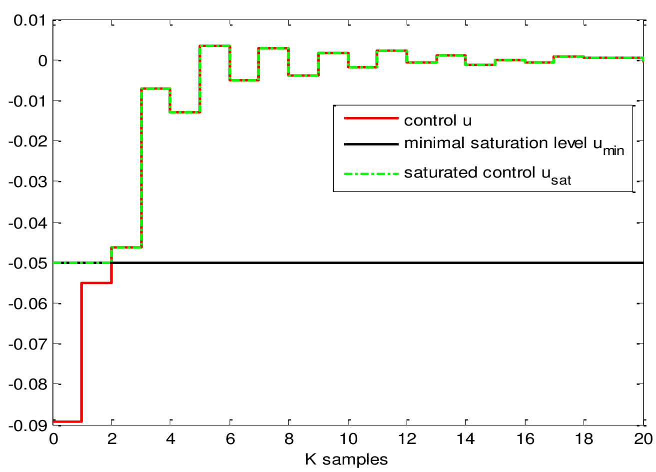

Figure 4. Control inputs.

The result demonstrated that, despite the saturation limits the designed controller guarantees the closed loop system convergence to the zero with a good perturbation reject.

V. CONCLUSION

In this paper, a non quadratic Lyapunov functions is used to obtain sufficient conditions of asymptotic stability for nonlinear uncertain discrete-time systems represented by T-S descriptor models subject input saturation and external disturbances. The main advantage of the proposed approaches is to synthesize the control law by considering the saturation limits while achieving a guaranteed $\mathcal{L}_2$-gain performance. A non PDC control law is used to achieve this objective. The controller gains are then obtained by solving an optimization problem under LMI constraints. Through two numerical examples the efficiency of the proposed techniques has been demonstrated.

REFERENCES

[1] K.Tanaka, T. Ikeda. and H.O.Wang, "Fuzzy regulators and fuzzy observers : relaxed stability conditions and LMI-based designs," *IEEE Transactions on Fuzzy Systems*, vol.6,pp:250-265,1998.

[2] K.Tanaka and H.O.Wang "Fuzzy Control system, design and Analysis: a Linear Matrix Inequality Approach," *Wiley-Interscience*, 2001.

[3] M. Johansson, A. Rantzer, and K.E. Arzen, "Piecewise quadratic stability of fuzzy systems", *IEEE Transactions on Fuzzy Systems*, vol. 7 (6), pp. 713-722, 1999.

[4] T.M. Guerra and L.Vermeiren., "LMI based relax non-quadratic stabilization for non-linear systems in the Takagi-Sugeno's form", *Automatica*, vol.40(5),pp.823-829, May 2004.

[5] T.M.Guerra, M. Bernal, K. Guelton, and S.Labiod, "Non-quadratic local stabilization for continous-time Takagi-Sugeno models", *Fuzzy Sets and Systems*, 2012.

[6] T.Taniguchi, K.Tanaka and H.O.Wang, " Fuzzy Descriptor system and nonlinear model following Control," *IEEE transactions on Fuzzy Systems*, vol.8(4),pp.442-452, 2000.

[7] V.E.Manzo, Z..Lendek,T.Guerra, and P.Pudlo, "Control design for discret-time descriptor models: a systematic LMI approach", *IEEE Trans.On Fuzzy Systems*, Early Access Online, 2015.

[8] A-T.Nguyen, M.Damrine,J.Lauber, "Simulataneous design parallel distributed output feedback and anti-windup compensators for constrained Takagi-Sugeno fuzzy systems," *Asian.J. Control*, vol.18,pp.1641-1654, 2006.

[9] A. Benzaouia,A. El hajjaji, and R.Oudah. "Fault tolerant saturated control for T-S fuzzy discrete-system with delay," *Nonlinear analysis:Hybrid systems*, vol.18, pp.60-71, 2015.

[10] D.Saifia, M.Chadli, S. Labiod, and T.M.Guerra, "Robust H∞ static output feedback stabilization of T-S fuzzy system subject t actuator saturatio," *Inter.J.Conf,Automation and System*, vol.10(3), pp.613-622, 2012.

[11] S. Bezzaoucha, B. Marks, D. Maquin, & J. Ragot, "Stabilization and output feedback control for Takagi-Sugeno with saturated actuators". *Inter.J.Adaptive control and signal Processing*. vol.30, pp.888-905, 2016.

[12] T.M. Guerra, V. Estrada-Manzo, and Z. Lendek, "Observer design for Takagi-Sugeno descriptor models: An LMI approach" *Automatica*, vol.52,pp.154-159, 2015.

[13] K. Guelton, S. Delprat, and T.M. Guerra, "An Alternative to inverse dynamics joint torques estimation in human stance based on a Takagi-Sugeno unknow-inputs observer in the descriptor form," *Control Engineering Practice,* vol.16(12),pp.1414-1426,2008.

[14] S. Aouaouda, M. Chadli, M. Boukhnifer, H.R. Karimi, "Robuste fault tracking controller design for vehicle dynamics: A descriptor appraoch", *Mechatronics*, vol. 30, pp. 316-326, 2015.

[15] M.Chadli and M. Darouach, "Novel bounded real lemma for discrete-time descriptor systems: Application to control-design," *Automatica*, vol.48, no.2, pp.449-453,Feb.2012.

[16] Q.V. Dang, L.Vermeiren, A. Dequidt, M. Dambrine, "Robust stabilizing controller design for Takagi-Sugeno fuzzy descriptor systems under state constraints and actuator saturation," *Fuzzy Sets and Systems*, vol. 329, pp. 77-90, 2017.